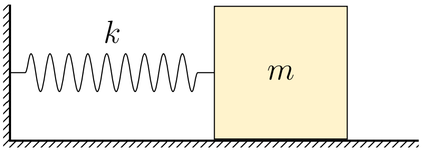
Chaos is merely order waiting to be deciphered
José Saramago, The Double.

Chaos

Introduction

Let's consider the oscillations of a harmonic oscillator.



Its equation of motion has the form of

$$m\ddot{x} = -kx.$$

This equation has an analytical solution

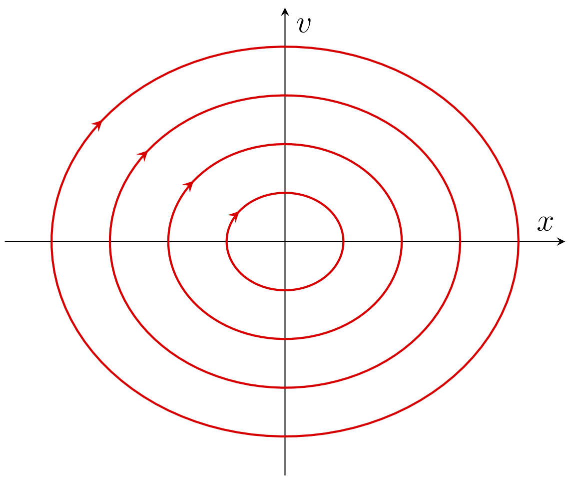
$$\begin{cases} x(t) = A\cos\left(\omega t + \varphi_0\right); \\ v(t) = -A\omega\sin\left(\omega t + \varphi_0\right), \end{cases}$$

where $\omega = \sqrt{k/m}$, and we can find the oscillation amplitude A and the initial phase φ_0 from the initial conditions.

We will call the dependence $\dot{x}(x)$ a phase portrait. In the case of a harmonic oscillator, it can be found from the equation

$$x^2 + v^2/\omega^2 = \text{const}.$$

We can also determine the constant value on the right side from the initial conditions. Dependence $v(x)$, which follows from the last equation, is shown in the figure.



This solution is analytical, but, unfortunately, for most mechanics problems such a solution cannot be obtained. In this case, numerical methods are usually used.

A programming environment for the problem

We've prepared a notebook in the google colab so you do not have to program anything.

Follow the link to the notebook (just in case, here it is again), go to your google account, click on the code execution in any cell (that is a play button), and you get the good news that this code was written by someone called Eliseev Maxim, you agree with this and the notebook is ready to use.

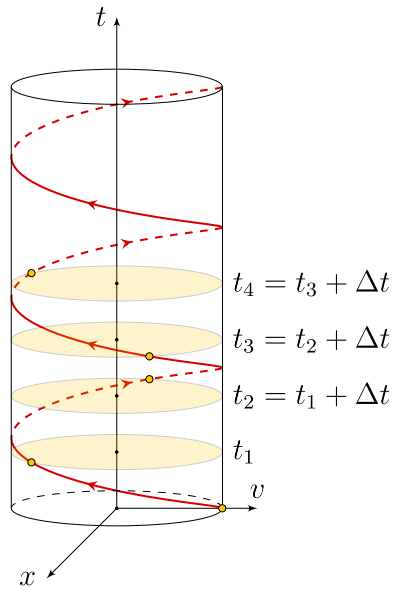
Details about how to use the notebook are described in this video (by the same Maxim Eliseev). ~~If~~ When you ~~still~~ will have questions about how to use this notebook, then you can ask this person in direct messages. Don't be shy, he is energetic and ready to answer 25 (twenty-five)/7.

Comment. You can use any other environment to solve this problem, but we do not guarantee that other numerical methods will lead to the correct solution. And there is a risk that we will not be able to help you with that in any way.

Poincaré section

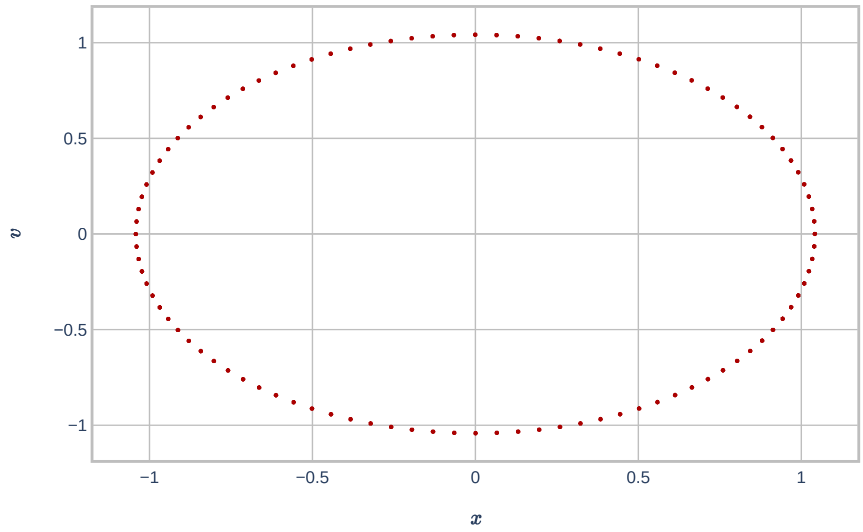
One of the most popular methods for numerically solving such differential equations is the Poincare section method. Let's demonstrate the operation how this method works on the example of a harmonic oscillator.

Let's depict the dependence $x(t)$, $v(t)$ in a three-dimensional form, where $x(t)$ and $v(t)$ are represented by the x and y axes, and the time t corresponds to the z-axis. In the case of a harmonic oscillator, we get a helix along the cylinder. At regular intervals, we will draw sections parallel to the plane $\{x, v\}$ and find the intersection point of the plane and the helix (see figure on the right), and then draw it on a separate plane $\{x, v\}$ (see figure below).



Note. It is clear that for a harmonic oscillator, a numerical solution method is not needed, since an analytical solution is possible. However, we see that the numerical method gives the same result as the analytical one. In more complex situations, the pictures will be different.

Note about code. The SolveSystem function has an argument ~~and you don't, lol~~ is_poincare_section. If you set it as is_poincare_section = false, then the sections will be located as tightly as possible (Δt won't be enough, this case is depicted in the figure with a phase portrait). If you set it as is_poincare_section = true, then the time interval will be significantly longer and in dimensionless variables will be equal to 2π.

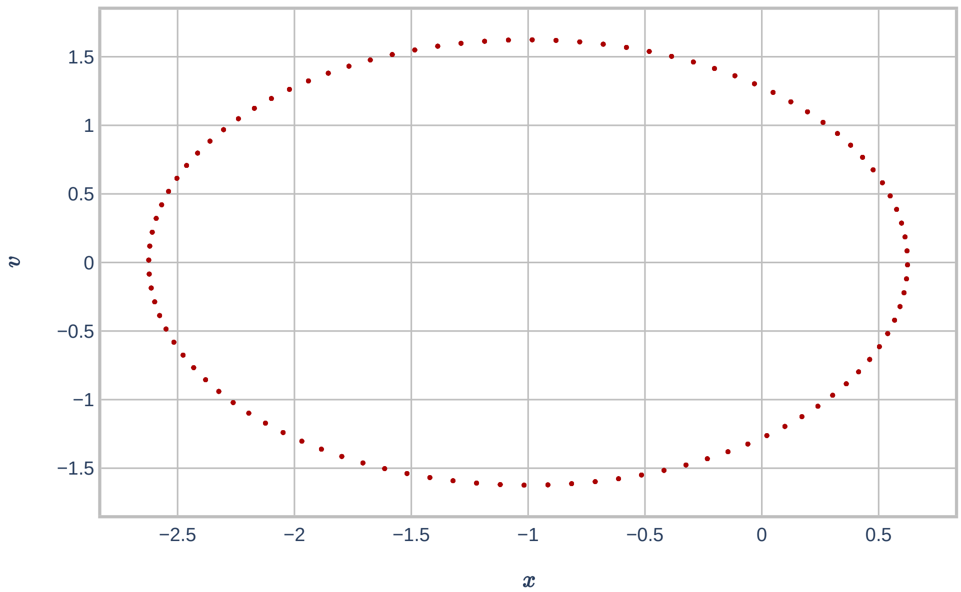


Look at a few more examples.

Example. Let a constant force act on the oscillator. In this case, the equation of motion will look like this

$$m\ddot{x} = -kx + F.$$

Try to change the correct parameters in the notebook to get the corresponding Poincaré section. It should be as shown in the figure.

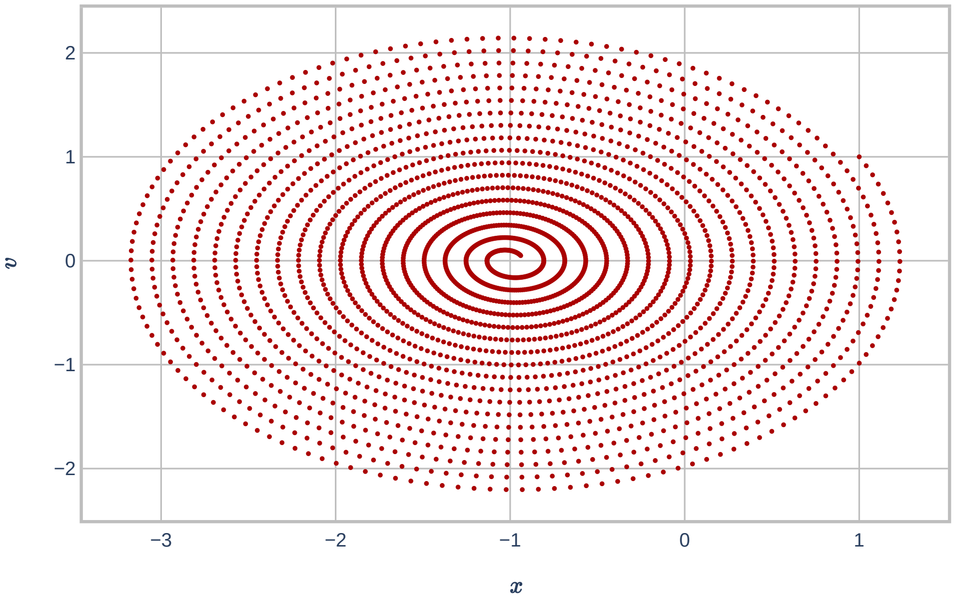


This is exactly the same ellipse as in the previous example, only its center is shifted by some amount, which corresponds to a shift in the equilibrium position. The direction of the shift depends on the force sign and the magnitude of the modulus F.

Example. Instead of a constant force, we add the dry friction force

$$m\ddot{x} = -kx - \mu mg \cdot \mathrm{sign}\left(\dot{x}\right).$$

In this case, the Poincare section will have the form of.



This example can be used to show that numerical methods do not always give results that correspond to reality and can produce various kinds of «artefacts». If it is unsuccessful to change the sampling step (time Δt), then in this case the Poincare section will not be constructed, or a deliberately absurd drawing will be obtained.

Numerical methods are a great tool, but they need to be carefully used. If you have questions about numerical methods, then you can ask here or there.

Part 1. Nonlinearity

The linear oscillator model does not always describe all the features of dynamical systems. For example, the model often works for small deviations from the equilibrium position. But if the deviations from the equilibrium position increase, then for this case the linear oscillator model should be clarified.

There are many different models of the nonlinear oscillator. We will consider some general dynamic model given by the following equation:

$$\ddot{x} + ax + bx^3 = 0. \tag{1}$$

1. *(1 point)* Find the potential energy $U(x)$ of such a system.
2. *(1 point)* Give examples of a system whose potential energy depends on the coordinate in a similar way (in terms of the number of minima and maxima) if

   (a) $a > 0, b > 0,$

   (b) $a > 0, b < 0,$

   (c) $a < 0, b > 0,$

   (d) $a < 0, b < 0.$

3. *(1,5 points)* For all the cases above, draw all possible phase portraits of the given system for various initial statements.

Let's add dissipation to the equation (1)

$$\ddot{x} + \gamma\dot{x} + ax + bx^3 = 0, \quad \gamma > 0. \tag{2}$$

4. *(0,5 points)* Draw all possible phase portraits of this system for various initial statements if $a < 0, b > 0$.
5. *(1 point)* Let $a = -2$, $b = 1$, $\gamma = 0{,}15$.
   (a) Find the equilibrium positions of the system.

   Let's assume that initial speed is $\dot{x}(0) = 10$, initial coordinnate $x(0) \in [-2{,}5; 2{,}5]$.

   (b) Find the value $x(0)$ such that the system comes to the equilibrium position on the left. Specify, to the nearest hundredth, the boundaries of the segment in which $x(0)$ lies, and in which the system comes from any point to this equilibrium position.

   (c) Find the value of $x(0)$ such that the system comes to the equilibrium position on the right. Specify, to the nearest hundredth, the boundaries of the segment in which $x(0)$ lies, and in which the system comes from any point to this equilibrium position.

Let's add an external harmonic force to the equation (2)

$$\ddot{x} + \gamma\dot{x} + ax + bx^3 = \varepsilon \sin t. \tag{3}$$

6. *(1 point)* Let $a = -1, b = 1$. Build the Poincare sections for the following parameters values:

   (a) $\gamma = 0$; $\varepsilon = 0$,
   (b) $\gamma = 0$; $\varepsilon = 0{,}005$,
   (c) $\gamma = 0$; $\varepsilon = 0{,}01$,
   (d) $\gamma = 0$; $\varepsilon = 0{,}1$,
   (e) $\gamma = 0{,}15$; $\varepsilon = 0{,}3$.

   Build them using both cases of the
7. *(0 points)* What conclusion can you make from the constructed Poincare sections?

Part 2. Discrete Model

It is often convenient to switch to discrete-time and observe the dynamics of a change in some quantity. For example, we can denote x_n as the population of a certain biological species at a time t_n (e.g., in a year with a number n). Then the equation describing the dynamics of population change in the simplest case can look like this

$$x_{n+1} = \lambda x_n,$$

where $\lambda = \text{const} > 0$ is a coefficient that determines the living conditions of a given species. It is clear that if $\lambda > 1$, then the population will grow indefinitely, if $\lambda = 1$, then its value will be constant from year to year, if $\lambda < 1$, then the population will die out. In this problem, we will analyze similar models.

We will denote the right side of the equation as $f(x_n)$. If some value x^* satisfies the condition $f(x^*) = x^*$, then we will call x^* as the equilibrium position.

If we apply our «function» twice, i. e. If we write an expression of the form $f(f(x))$, then we call this transformation quadratic and denote it as $f^2(x)$. For applying the function n times, we will use the notation $f^n(x)$. It is clear that the factor, in this case, is not equivalent to the concept of a factor from algebra.

Path of the Point. Linear case

Review in the notebook the cell «Path of the Point». It contains a «function»

$$f(x) = \lambda \cdot \min\left[(1-x), x\right], \quad x_{n+1} = f(x_n)$$

where $\lambda = \text{const} > 0$. In all parts of this section of the problem, we will consider only $x \in [0, 1]$.

7. *(0,2 points)* Within what limits can the parameter λ change so that x_n for any n belongs to the segment $[0, 1]$?
8. *(0 points)* Plot $f(x)$.
9. *(0,2 points)* Qualitatively plot $f^2(x)$ and $f^4(x)$ for values of the parameter $\lambda \neq 0$.
10. *(0,2 points)* Find the dependence of the number of equilibrium positions on the parameter λ of the system described by the «function» $f(x)$. For those values of λ, where the number of equilibrium positions is the greatest, find the dependence of the equilibrium positions $x^*(\lambda)$.
11. *(0,1 points)* Find the dependence of the number of equilibrium positions on the parameter λ of the system described by the «function» $f^2(x)$. For any parameter λ, where the number of equilibrium positions is maximum, develop a graphical method of finding those equilibrium positions.
12. *(0,3 points)* What is the maximum finite number of equilibrium positions for the system described by the «function» $f^n(x)$?
13. *(0,5 points)* For the «function» $f(x)$ and $\lambda = 1{,}5$, find the sequence x_n, if $x_0 = 0{,}6$, $x_0 = 0{,}4$, and $x_0 = 0{,}139$. Show the result in the form of a graph, which will display $f(x)$, as well as $g(x) = x$, or in the form of the first forty values of the sequence $x(n)$. Explain the result.
14. *(0,5 points)* For the «function» $f^2(x)$ and $\lambda = 1{,}5$, find the sequence x_n, if $x_0 = 0{,}6$, $x_0 = 0{,}61$. Show the result in the form of a graph, which will display $f(x)$, as well as $g(x) = x$, or in the form of the first forty values of the sequence $x(n)$. Try to find (or depict) the dependencies for other initial values of x.
15. *(0,5 points)* Open the cell of the notebook that is called «Discrete model. Linear case». It is able to build the dependence of the values of the points corresponding to the equilibrium positions for the function $f(x)$ on the parameter λ. Set the maximum value of λ and plot this graph. Describe the result. What features of the graph can you highlight? List all the properties you can find.

Path of the Point. Quadratic case.

Modifying the code in the cell «Path of the Point», analyze the following «function»

$$f(x) = \lambda x(1-x),$$

where $\lambda = \text{const} > 0$.

Note. This «function» can be found in various branches of science. From diffusion to economics.

In all parts of this section of the problem, we will consider only $x \in [0, 1]$.

16. *(0,2 points)* Within what limits can the parameter λ change so that x_n for any n belongs to the segment $[0, 1]$?
17. *(0,2 points) (0,2 points)* Plot $f(x)$. Find at which λ the number of equilibrium positions is the largest and find how this equilibrium position depends on λ.
18. *(0,2 points)* Plot $f^2(x)$ for any value of the parameter $\lambda \neq 0$.
19. *(0,2 points)* For the «function» $f(x)$ and $\lambda = 2$, find the sequence x_n, if $x_0 = 0{,}5$, $x_0 = 0{,}4$, and $x_0 = 0{,}33$. Show the result in the form of a graph, which will display $f(x)$, as well as $g(x) = x$, or in the form of the first forty values of the sequence $x(n)$. Explain the result.
20. *(0,2 points)* For the «function» $f^2(x)$ and $\lambda = 2$, find the sequence x_n, if $x_0 = 0{,}1$, $x_0 = 0{,}16$. Show the result in the form of a graph, which will display $f(x)$, as well as $g(x) = x$, or in the form of the first forty values of the sequence $x(n)$. Try to find (or depict) dependencies for other initial values x. Explain the result.
21. *(0,5 points)* Open the cell of the notebook that is called the «Discrete model. Quadratic case». It is able to build the dependence of the values of the points corresponding to the equilibrium positions for the function $f(x)$ on the parameter λ. Set the maximum value of λ and plot this graph. Describe the result. What features of the graph can you highlight? List all the properties you can find.

First hint — 16.05.2022 14:00 (Moscow time)
Second hint — 18.05.2022 14:00 (Moscow time)

Final of the fifth round — 20.05.2022 22:00 (Moscow time)