LPR VI Cup

9.s06.e05

The ending is not as important as the moments that lead up to it
To the Moon

Contents

Common

In the Second Episode for grades 9-10, we talked with the LPR Cup Participants about time, and in grade 11 we offered a problem that could be elegantly solved using a geometric method. In the Third Episode for grades 10-11, Participants were introduced to the variational principle and the use of symmetry to analyze the propagation of light in gradient media.

~~This bzzzz is not for nothing~~ All of this was not for nothing.

When studying objects that move at high speeds (at speeds comparable to the speed of light), the classical laws of Newtonian mechanics become inapplicable and yield results that contradict experiment. To describe the motion of such objects, a separate branch of physics was developed, called the *special relativity theory*. As experimental data continued to accumulate, particularly in the study of gravity, it became clear that the theory could be developed further, leading to the creation of the *general relativity theory*.

These branches of physics speak the language of geometry and use concepts such as: metric, interval, curvature, and so on. In addition, these fields use the concepts of action and the variational principle.

In this Episode, we will tell you about this special language, how it works, and what results can already be obtained and understood with its help.

In section 1, we will discuss issues related to the special theory of relativity (SRT), after which we will see that SRT is not sufficient to describe objects such as black holes (section 2), and we will invite Participants in grades 10-11 to move on to the general relativity theory.

Good luck!

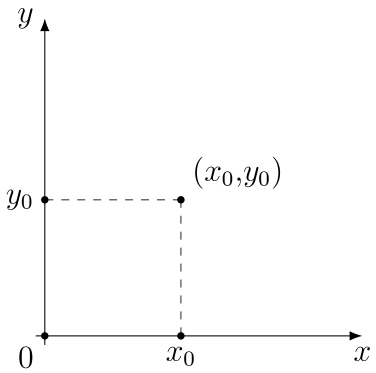
Charles the Cat

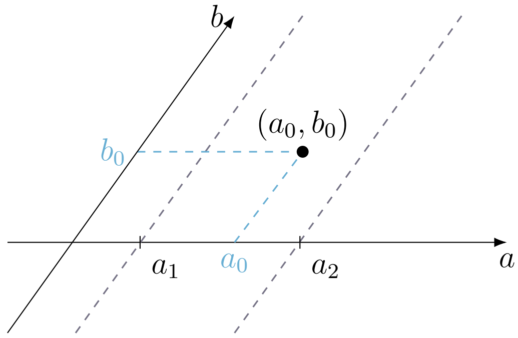
1 Kinematics of SRT

The special relativity theory (SRT) is a physical theory that has found a huge number of practical applications and actively uses the language of geometry — specifically, the geometry of Minkowski spacetime. In this section, we will study the kinematic aspects of SRT.

Calculating Distances in Different Coordinates

Let us consider a two-dimensional Cartesian coordinate system. In this case, the position of points is specified by a pair of numbers (x, y). To find the coordinate of a point along one of the axes by construction, we need to draw a straight line through this point parallel to the other axis until it intersects the first axis. This algorithm is general and also works in the case where the axes are not perpendicular to each other. It also follows that the lines of constant coordinate along one axis are straight lines parallel to the other axis.





A trajectory in such a coordinate system is defined as the dependence of one coordinate on another, for example, $y(x)$. The distance in such a coordinate system is determined using the Pythagorean theorem, that is, a small element of length is written as:

$$ds^2 = dx^2 + dy^2.$$

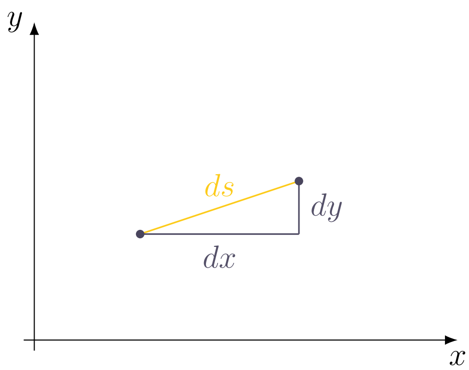
Note that in the case of non-perpendicular axes, the distance is determined by the law of cosines:

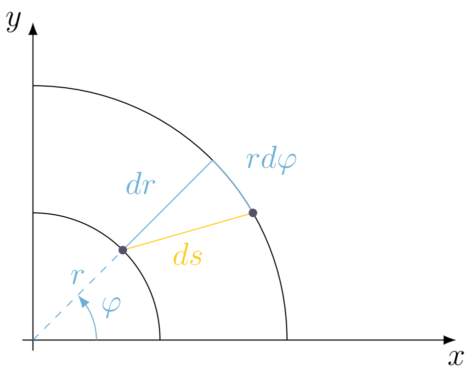
$$ds^2 = da^2 + db^2 + 2da db \cos\alpha,$$

where α is the angle between the axes a and b.

The Cartesian coordinate system is not the only way to specify the position of a point on a plane. There are many other options, for example, polar coordinates, in which the position of a point is given by the distance from the origin r and the angle φ measured from some axis (see the figure). When switching to polar coordinates, the distance takes the form:

$$ds^2 = dr^2 + r^2 d\varphi^2.$$





Postulates of SRT

The special relativity theory of is based on the following postulates:

1. The laws of physics are the same in all inertial reference frames.

2. The speed of light in vacuum is constant.

The first postulate is straightforward—we assume that processes in all inertial reference frames (IRFs) occur in the same way, and therefore the physical laws are written in the same form.

The second postulate, however, seems quite surprising, but it is precisely this postulate that serves as the starting point for constructing SRT. The fact that the speed of light is constant was experimentally confirmed in various experiments conducted in the second half of the 19th century (the Fizeau experiment, the Michelson–Morley experiment, and others).

Galilean and Lorentz Transformations

In this section, we will consider the transition from one inertial reference frame K to another inertial reference frame K'.

Let us consider the motion of a material point along the x-axis with velocity v. When moving from the laboratory reference frame to the reference frame of this point, the coordinates and time are transformed according to the law

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t.$$

These transformations are called Galilean transformations. From them, it follows that the velocities in this reference frame are

$$v'_x = \frac{dx'}{dt'} = v_x - v, \quad v'_y = \frac{dy'}{dt'} = v_y, \quad v'_z = \frac{dz'}{dt} = v_z.$$

Such a law of velocity transformation contradicts the second postulate of special relativity. Let us confirm this with a classic example. Imagine a person running with a flashlight in hand. In the reference frame associated with the running person, photons (or the electromagnetic wave) have a speed equal to the speed of light in vacuum (hereafter denoted as c, whose numerical value we will take to be approximately $3 \cdot 10^8$ m/s). In the reference frame associated with a stationary observer, according to the Galilean transformations, light should have a speed of $c + u$, where u is the speed of the running person, which contradicts the second postulate of special relativity, according to which this speed must be equal to c.

Let us find the transformation of coordinates and time under which the speed of light does not change when moving from one inertial reference frame to another (for convenience, we will call these frames K and K'). Since in any inertial reference frame the dependence $x(t)$ for a point moving at constant speed is linear, these transformations must map straight lines to straight lines, i.e., they must be linear. In the general case, such transformations have the form:

$$\begin{cases} x' = Ax + Bt, \\ t' = Cx + Dt. \end{cases}$$

1.1. *(0 points)* Using the fact that the dependence of the photon's coordinate on time in two different inertial reference frames K and K' is written as $x = ct$ and $x' = ct'$, find the explicit form of the transformations that preserve the speed of light (these are called Lorentz transformations):

$$\begin{cases} x' = \gamma\left(x - vt\right), \\ t' = \gamma\left(t - \dfrac{v}{c^2}x\right), \end{cases}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ is the relativistic "gamma factor" and v — velocity of K' in relatively to K.

Note 1. In this example, we considered motion only along the x-axis, so the coordinates along the y and z axes will have the form: $y' = y$, $z' = z$.

Note 2. It is clear that in the classical (non-relativistic) limit, when $v/c \ll 1$, the Lorentz transformations coincide with the Galilean transformations.

Note 3. From the expression for the "gamma factor," it follows that a reference frame moving at a speed greater than the speed of light is not acceptable in this theory and has no physical meaning. Thus, we can conclude that the speed of light in vacuum is not just a constant, but the maximum allowed speed for any particle.

Using the fact that when transforming back from K' to K only the sign of the velocity v changes, it is clear that the inverse Lorentz transformation has the form:

$$\begin{cases} x = \gamma\left(x' + vt'\right), \\ t = \gamma\left(t' + \dfrac{v}{c^2}x'\right), \end{cases}$$

1.2. *(0 points)* Category Hobbit There and Back Again. Make sure that by consecutively applying two Lorentz transformations for the transitions $K \to K' \to K$, we obtain the identity transformation for coordinates and time.

Let us consider the motion of the same material point along the x-axis in two inertial reference frames K and K'. Writing the Lorentz transformation for two moments in time and subtracting one system of equations from the other, we get:

$$\begin{cases} dx' = \gamma\left(dx - vdt\right), \\ dt' = \gamma\left(dt - \dfrac{v}{c^2}dx\right), \\ dy' = dy, \\ dz' = dz. \end{cases}$$

Minkowski Spacetime

Now that we have established that under Lorentz transformations, time—just like spatial coordinates—transforms nontrivially, it makes sense to combine them into a single Minkowski spacetime, where the coordinates of points are given by (ct, x, y, z).

1.3. *(0.4 points)* Draw and describe how, under Lorentz transformations for motion only along the x-axis, the coordinate axes ct and x change.

In Minkowski spacetime, each point is an *event* that occurs at a certain spatial coordinate at a certain moment in time. The distance between two events in Minkowski spacetime is called the *interval* and is calculated as follows:

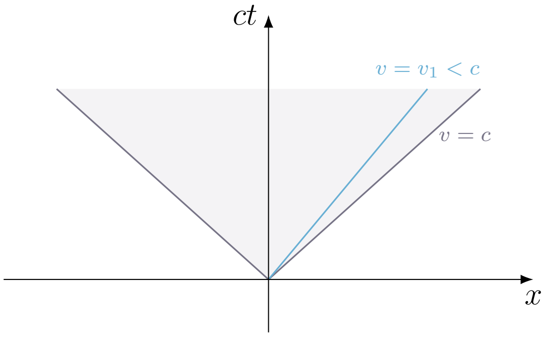
$$ds^2 = g_{tt}c^2dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

The set of coefficients g_{tt}, g_{xx}, g_{yy}, g_{zz} is called the *metric.*

1.4. *(0.4 points)* Show that Lorentz transformations do not change the interval of Minkowski spacetime (such transformations are called isometries of the metric).

The square of the interval between two events can have any sign. Let us consider the events (0,0) and (ct, x) and the interval ds between them.

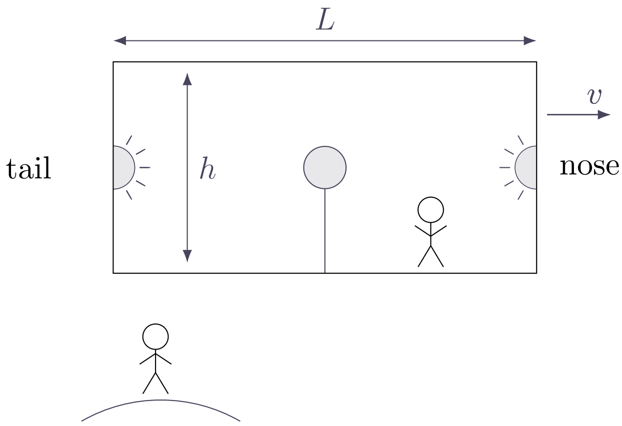
- Events for which $ds^2 = 0$ lie on the light cone. These events correspond to a photon emitted from $(0,0)$ arriving at the point with spatial coordinate x at time t. Such an interval ds is called *lightlike* (or *null*).
- Events for which $ds^2 < 0$ lie outside the light cone. Such an interval is called *spacelike*.
- Events for which $ds^2 > 0$ lie inside the light cone. Such an interval is called *timelike*. Particles in Minkowski spacetime move along curves for which, for any small segment, $ds^2 > 0$. Such curves are called *worldlines*.



Effects of Relativistic Kinematics

Let us consider a few examples that will demonstrate the peculiarities of Lorentz transformations.

1.5. *(0.6 points)* Suppose a rocket is flying through outer space at a speed $v = c/2$. The interior cabin of the rocket has the shape of a rectangular parallelepiped, with light sources installed at the centers of the "nose" and "tail." Short flashes from these sources occur simultaneously according to the clock of an astronaut inside the rocket. Find the difference between the detection times of the flashes by a receiver located at the center of the cabin, as measured in the reference frame of an observer on a stationary planet. In the same reference frame, find the difference between the emission times of flashes by the sources. The distance from the "nose" to the "tail" in the reference frame of a rocket is L.



1.6. *(0.6 points)* Depict the events of light detection and emission in the reference frames of an astronaut and observer on the planet using Minkowski spacetime coordinates.

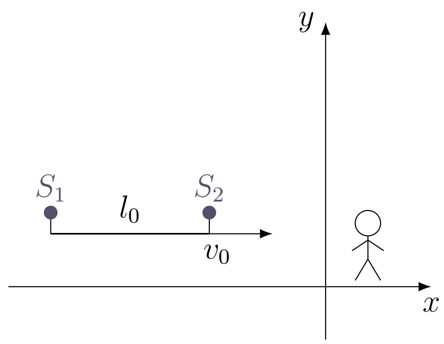
1.7. *(1 points)* For these two observers, how are the times related for the light emitted from the "tail" source to first reach the "floor" of the rocket, if the distance from the source to the ceiling is h?

Now let us consider a real-life example from the world of particle physics.

1.8. *(0.6 points)* A meson is produced in the atmosphere at an altitude of about 10 km above the Earth's surface. Its proper lifetime (proper quantities for a given object are those measured in its own rest frame) is approximately 2,2 μs. Estimate what speed the meson must have in order to reach a detector on the Earth's surface.

And now a few more abstract examples.

1.9. *(1 points)* A rod has a proper length l_0. Two light bulbs, S_1 and S_2, are attached to the ends of the rod. The rod moves with velocity v_0 toward a stationary observer. Bulb S_1 emits light earlier than S_2, so that both flashes reach the observer simultaneously. At the moments the light is emitted, bulbs S_1 and S_2 are located at points x_1 and x_2, respectively. What distance $x_1 - x_2$ between the bulbs will the observer measure? (This will be the apparent length of the rod, as perceived by the human eye or recorded by a camera.)



1.10. *(0.8 points)* From the origin of the inertial reference frame K, short light pulses are sent along the x-axis at intervals of time T (according to the clocks in K). Find the time interval between the moments these signals are registered by an observer if the observer is in the inertial reference frame K', which is moving towards system K with velocity $v = c/2$.

The Twin Paradox

1.11. *(0.8 points)* A spaceship leaves Earth at speed βc at time $t_0 < 0$ (by earthly clocks), where β is a known constant. The captain's twin brother remains on Earth. At time $t_0/2$ by earthly clocks, the spaceship makes a quick turnaround and returns to Earth at the same speed as before. Analyze the astronaut's journey in the reference frame of the Earth observer and show that the astronaut's proper time elapsed during the trip is less than the proper time of his brother (in other words, the twin astronaut has aged less).

Note that similar reasoning can be made from the point of view of the traveling brother, who considers himself at rest and the Earth observer as moving relative to him at the same speed. In this case, it would seem that the brother on Earth has aged less. Explain qualitatively how to resolve this paradox, and approximately what the actual difference in proper times of the two twins will be (indicate which one will be older at the end of the journey).

Velocity Addition

Let us consider the transformation of velocities in special relativity.

1.12. *(0 points)* Suppose a certain object moves with velocity v along the x-axis in an inertial reference frame associated with a stationary observer. A rocket is moving towards it with a relativistic velocity u. Find the velocity of the object in the reference frame associated with the rocket.

1.13. *(0 points)* What will the answer look like if the rocket is moving in the same direction as the observed object?

1.14. *(0 points)* Check that if the velocity $v = c$, the velocity does not change.

1.15. *(0.8 points)* Two rods, each with proper length l_0, move towards each other parallel to a common axis with relativistic velocities. An observer associated with one of them records that the time between the coincidences of the left and right ends of the rods is T. What is the relative velocity of the rods?

2 Classical black hole

In the previous section, you became acquainted with the postulates of the Special Relativity Theory. It turns out that one of its direct consequences is the possibility of the existence of special objects—black holes—which inevitably "absorb" any matter that comes sufficiently close to them. In this section, you are invited to conduct your own investigation of such an object.

Through prolonged observation of the position of a star near the center of the galaxy, it was found that the star undergoes periodic motion in the gravitational field of a certain massive object. Assume that the distance from the observation point to the center of the galaxy is approximately $26 \cdot 10^3$ light-years, the period of the star's orbit along its trajectory is approximately 16 years, and the plane in which the star moves is perpendicular to the line of sight.

2.1. *(0.5 points)* Using the experimental data, determine the position of the object attracting the star in the coordinates provided in the problem.

2.2. *(1 point)* Using the experimental data, determine the mass of the object attracting the star.

2.3. *(1.5 points)* Assuming the object attracting the star is spherically symmetric and its size is sufficiently small, determine the boundaries of the region around such an object from which no signal can reach a distant observer.

Table 1: Coordinates of the star observation

$x,'' \cdot 10^3$	−70	−60	−44	−27	−10	9	31	51	72	87	95
$y,'' \cdot 10^3$	133	150	166	177	182	185	182	174	159	139	118

The experimental data are given in an orthogonal coordinate system, where each axis is measured in arcseconds ($''$) and represents the angular deviation of a point from the origin.

You are allowed to use numerical methods to analyze the experimental data, and the parameters determined from them must be obtained with an accuracy of $5 \cdot 10^{-3}('')$.

First hint — 28.05.2025 20:00 (Moscow time)
Second hint — 30.05.2025 12:00 (Moscow time)

Final of the fifth round — 01.05.2025 18:00 (Moscow time)