LPR Cup

9.s03.e05

Prepare for complications
House M.D.

Chaos and Discrete Model

In this problem, you have to apply both analytical computing skills and numerical modeling skills. It's not as scary as it might seem at first.

A programming environment for the problem

We've prepared a notebook in the google colab so you do not have to program anything.

Follow the link to the notebook (just in case, here it is again), go to your google account, click on the code execution in any cell (that is a play button), and you get the good news that this code was written by someone called Eliseev Maxim, you agree with this and the notebook is ready to use.

Details about how to use the notebook are described in this video (by the same Maxim Eliseev). ~~If~~ When you ~~still~~ will have questions about how to use this notebook, then you can ask this person in direct messages. Don't be shy, he is energetic and ready to answer 25 (twenty-five)/7.

Comment. Most part of the notebook is learning, which you are not suppoosed to skip, and the task of 10-11 grades, which you can skip (at least not). The main work will be in the section «Discrete Model» and «Path of the Point».

Comment. You can use any other environment to solve this problem, but we do not guarantee that other numerical methods will lead to the correct solution. And there is a risk that we will not be able to help you with that in any way.

Discrete Model

It is often convenient to switch to discrete-time and observe the dynamics of a change in some quantity. For example, we can denote x_n as the population of a certain biological species at a time t_n (e.g., in a year with a number n). Then the equation describing the dynamics of population change in the simplest case can look like this

$$x_{n+1} = \lambda x_n,$$

where $\lambda = \text{const} > 0$ is a coefficient that determines the living conditions of a given species. It is clear that if $\lambda > 1$, then the population will grow indefinitely, if $\lambda = 1$, then its value will be constant from year to year, if $\lambda < 1$, then the population will die out. In this problem, we will analyze similar models.

We will denote the right side of the equation as $f(x_n)$. If some value x^* satisfies the condition $f(x^*) = x^*$, then we will call x^* as the equilibrium position.

If we apply our «function» twice, i. e. If we write an expression of the form $f(f(x))$, then we call this transformation quadratic and denote it as $f^2(x)$. For applying the function n times, we will use the notation $f^n(x)$. It is clear that the factor, in this case, is not equivalent to the concept of a factor from algebra.

Path of the Point. Linear case

Review in the notebook the cell «Path of the Point». It contains a «function»

$$f(x) = \lambda \cdot \min\left[(1-x), x\right], \quad x_{n+1} = f(x_n)$$

where $\lambda = \text{const} > 0$. In all parts of this section of the problem, we will consider only $x \in [0,1]$.

1. *(0,5 points)* Within what limits can the parameter λ change so that x_n for any n belongs to the segment $[0,1]$?
2. *(0 points)* Plot $f(x)$.
3. *(0,5 points)* Qualitatively plot $f^2(x)$ and $f^4(x)$ for values of the parameter $\lambda \neq 0$.
4. *(0,5 points)* Find the dependence of the number of equilibrium positions on the parameter λ of the system described by the «function» $f(x)$. For those values of λ, where the number of equilibrium positions is the greatest, find the dependence of the equilibrium positions $x^*(\lambda)$.
5. *(0,3 points)* Find the dependence of the number of equilibrium positions on the parameter λ of the system described by the «function» $f^2(x)$. For any parameter λ, where the number of equilibrium positions is maximum, develop a graphical method of finding those equilibrium positions.
6. *(0,7 points)* What is the maximum finite number of equilibrium positions for the system described by the «function» $f^n(x)$?
7. *(1,3 points)* For the «function» $f(x)$ and $\lambda = 1{,}5$, find the sequence x_n, if $x_0 = 0{,}6$, $x_0 = 0{,}4$, and $x_0 = 0{,}139$. Show the result in the form of a graph, which will display $f(x)$, as well as $g(x) = x$, or in the form of the first forty values of the sequence $x(n)$. Explain the result.
8. *(1,7 points)* For the «function» $f^2(x)$ and $\lambda = 1{,}5$, find the sequence x_n, if $x_0 = 0{,}6$, $x_0 = 0{,}61$. Show the result in the form of a graph, which will display $f(x)$, as well as $g(x) = x$, or in the form of the first forty values of the sequence $x(n)$. Try to find (or depict) the dependencies for other initial values of x.
9. *(1 point)* Open the cell of the notebook that is called «Discrete model. Linear case». It is able to build the dependence of the values of the points corresponding to the equilibrium positions for the function $f^n(x)$ on the parameter λ. Set the maximum value of λ and plot this graph. Describe the result. What features of the graph can you highlight? List all the properties you can find.

Path of the Point. Quadratic case.

Modifying the code in the cell «Path of the Point», analyze the following «function»

$$f(x) = \lambda x(1-x),$$

where $\lambda = \text{const} > 0$.

Note. This «function» can be found in various branches of science. From diffusion to economics.

In all parts of this section of the problem, we will consider only $x \in [0, 1]$.

10. *(0,5 points)* Within what limits can the parameter λ change so that x_n for any n belongs to the segment $[0, 1]$?
11. *(0,5 points) (0,2 points)* Plot $f(x)$. Find at which λ the number of equilibrium positions is the largest and find how this equilibrium position depends on λ.
12. *(0,5 points)* Plot $f^2(x)$ for any value of the parameter $\lambda \neq 0$.
13. *(0,5 points)* For the «function» $f(x)$ and $\lambda = 2$, find the sequence x_n, if $x_0 = 0{,}5$, $x_0 = 0{,}4$, and $x_0 = 0{,}33$. Show the result in the form of a graph, which will display $f(x)$, as well as $g(x) = x$, or in the form of the first forty values of the sequence $x(n)$. Explain the result.
14. *(0,5 points)* For the «function» $f^2(x)$ and $\lambda = 2$, find the sequence x_n, if $x_0 = 0{,}1$, $x_0 = 0{,}16$. Show the result in the form of a graph, which will display $f(x)$, as well as $g(x) = x$, or in the form of the first forty values of the sequence $x(n)$. Try to find (or depict) dependencies for other initial values x. Explain the result.
15. *(1 point)* Open the cell of the notebook that is called the «Discrete model. Quadratic case». It is able to build the dependence of the values of the points corresponding to the equilibrium positions for the function $f(x)$ on the parameter λ. Set the maximum value of λ and plot this graph. Describe the result. What features of the graph can you highlight? List all the properties you can find.

First hint — 16.05.2022 14:00 (Moscow time)
Second hint — 18.05.2022 14:00 (Moscow time)

Final of the fifth round — 20.05.2022 22:00 (Moscow time)