LPR VI Cup

11.s06.e05

Hint 2

IMPORTANT! The next task is both a hint and an alternative to the main task. Three important points:

1. You can continue to send the solution to the main problem.
2. At any moment before the final deadline you can start to solve the Alternative problem. If you do so, at the beginning of the solution write: *I am doing the Alternative problem!* In this case a penalty coefficient for the Alternative problem is
$$0{,}7 \cdot \sum_i \frac{k_i \cdot p_i}{10},$$
where p_i is a point for the problem item, and k_i is a penalty coefficient for the corresponding problem's item at the moment of moving to the Alternative problem. In other words, maximal points for the alternative problem equals to the maximal points you can gain at the moment of moving to the alternative one multiplied by 0,7. Also, we remind you that a penalty coefficient can't be less than 0,1. Solutions of the main problems from that moment will not be checked. Be careful!
3. The task consists of several items. The penalty multiplier earned **before** is applied to all points. In the future, each item is evaluated as a separate task. If you send a solution without any item, this item's solution is considered as Incorrect. For more information about scoring points for composite tasks, see the rules of the Cup.

Contents

Alternative task

1 Kinematics of SRT

1.1. Two events occur in the laboratory frame of reference in the same place, but they are separated in time by 3 seconds.

   (a) *(0.2 points)* What is the distance in space between these events in the rocket's reference frame, if the time interval between events is 5 seconds in it?

   (b) *(0.2 points)* What is the velocity v_r of the rocket relative to the laboratory reference frame?

1.2. *(0.4 points)* A spaceship moves at a constant speed $V = (24/25)c$ toward the center of the Earth. What distance, in the reference frame associated with the Earth, will the spaceship travel during a time interval $\Delta t' = 7$ s as measured by the ship's clock? Neglect the rotation of the Earth and its orbital motion.

1.3. *(0.6 points)* A spaceship is flying at a speed of $V = 0.6\,c$ from one space beacon to another. At the moment when it is exactly halfway between the beacons, each beacon emits a light pulse toward the ship. Find the time interval, as measured on the ship, between the moments when these pulses are detected. The distance between the beacons is such that light takes 2 months to travel from one to the other.

1.4. *(0.8 points)* Two starships with their engines off are moving toward each other. On one starship, signal lights flash simultaneously at the bow and stern every second. On the approaching starship, every 0.5 seconds, two flashes are observed with a time interval of $\tau' = 1$ μs. Find the length l_0 of the first starship and their relative approach speed v.

Newton's second law in SRT has the form

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad \text{где} \quad \mathbf{p} = \gamma m v.$$

1.5. *(0.8 points)* A particle with a mass of m begins to move under the influence of a constant magnitude and direction of force F. Determine after what time, according to its own clock, the particle will reach a velocity of $v = 0.8c$. The following integral may be useful to you:

$$\int\limits_{x_0}^{x_1} \frac{dx}{1 - x^2} = \frac{1}{2} \ln \frac{1 + x_1}{1 - x_1} \frac{1 - x_0}{1 + x_0}.$$

2 Classical black hole

During prolonged observation of the position of a star near the center of the galaxy, it was found that the star undergoes periodic motion along a circular path in the gravitational field of a certain massive object. It is known that the distance from the observation point to the center of the galaxy is $\approx 26 \cdot 10^3$ light-years, the period of the star's orbit is ≈ 16 years, the radius of the star's circular trajectory as seen from the observation point is visible at an angle of $88 \cdot 10^{-3}$ arcseconds, and the plane in which the star moves is perpendicular to the direction of observation.

2.1. *(1 point)* Determine the mass of the object attracting the star.

2.2. *(1 point)* Assuming the object attracting the star is spherically symmetric and its size is sufficiently small, determine the boundaries of the region around such an object from which no signal can reach a distant observer.

3 GRT

Light rays in the Schwarzschild metric

Let's consider how light moves in the Schwarzschild metric.

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\varphi^2. \tag{1}$$

Far from gravitating bodies, light propagates in a straight line. The distance from this line (in (x, y, z) space) to the origin is called the *impact parameter* ρ. Since the mass of the photon is zero, far from gravitating bodies its energy is $E = |\mathbf{p}|$, and its angular momentum is $J = |\mathbf{p}|\rho$.

Recall that the curve along which light moves is a null (lightlike) geodesic, i.e., at each of its points

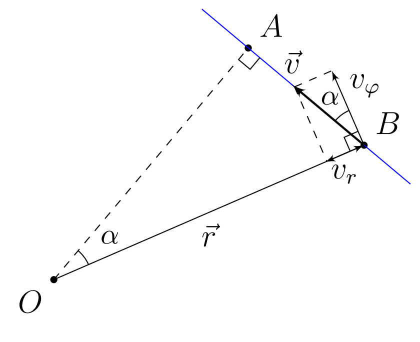
$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\varphi^2 = 0. \tag{2}$$

Due to the fact that this equation is fulfilled, a light-like geodesic depends only on a single parameter ρ instead of the two parameters E and J. To obtain $t(r, \rho)$ and $\varphi(r, \rho)$, we need one more equation. Let us take, as such an equation, the ratio of the conservation laws obtained in the main problem:

$$\rho = \frac{J}{E} = \left(1 - \frac{r_g}{r}\right)^{-1} r^2 \frac{d\varphi}{dt} = \text{const}\,. \tag{3}$$

It is clear that for $r \gg r_g$ this reduces to rv_φ, which coincides with the distance described above:

$$OA = r\cos\alpha = r\frac{v_\varphi}{|\vec{v}|} = r\frac{v_\varphi}{c} = rv_\varphi.$$



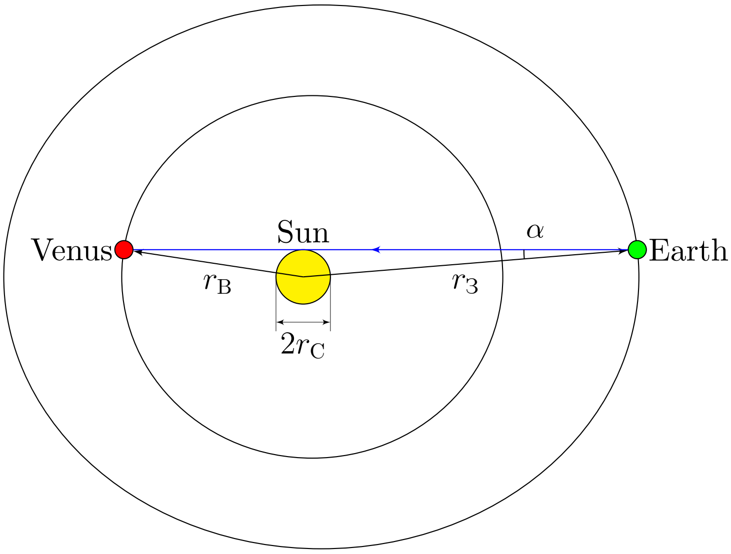
Shapiro delay

One of the consequences of light propagating in the metric 1 is that its propagation speed decreases, and therefore the travel time increases. This effect was predicted by Irwin Shapiro in 1964 and was later experimentally confirmed in 1966–1967. In this section, you are invited to reproduce Shapiro's calculation.

Let us consider a light ray emitted from Earth and reaching Venus. For simplicity, we will assume that the trajectory of the ray is a straight line[1], i.e., that equation (3) has the form

$$\rho = r\frac{v_\varphi}{|\vec{v}|} = r\frac{r d\varphi/dt}{\sqrt{(dr/dt)^2 + r^2(d\varphi/dt)^2}} = r\frac{r d\varphi}{\sqrt{dr^2 + r^2 d\varphi^2}} = r\sin\alpha = \text{const}\,.$$

To maximize the effect, consider the trajectory that touches the Sun (see the picture below).



Let the distance from the Sun to Earth be r_E, and to Venus be r_V in this planetary configuration. The radius of the Sun is r_S.

3.1. *(0.5 points)* Express dt for such a trajectory in terms of dr, r, r_S, and the Schwarzschild radius r_g.

3.2. *(0.5 points)* Decompose the resulting expression for dt to the first order of smallness in r_g/r.

3.3. *(0.5 points)* Using the decomposition obtained in the previous paragraph, find the time t_{gr} it takes for the ray to reach Venus. The following integrals may be useful to you:

$$\int_{x_0}^{x_1} \frac{dx}{\sqrt{x^2 - a^2}} = \ln\frac{x_1 + \sqrt{x_1^2 - a^2}}{x_0 + \sqrt{x_0^2 - a^2}}, \qquad \int_{x_0}^{x_1} \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x_1^2 - a^2}}{a^2 x_1} - \frac{\sqrt{x_0^2 - a^2}}{a^2 x_0}.$$

3.4. *(0.5 points)* Let us denote the Newtonian gravity prediction for the time required for the ray to reach Venus as t_N. Assuming $r_S = 7 \times 10^5$ km, $r_E = 1.5 \times 10^8$ km, $r_V = 1.1 \times 10^8$ km, and the Schwarzschild radius for the Sun is $r_g = 3$ km, calculate $t_{gr} - t_N$.

Deflection of light rays by a black hole

From equations (2) and (3), it is possible to obtain the equations for the light-like geodesic $t(r,\rho)$ and $\varphi(r,\rho)$:

$$t(r,\rho) - t_0 = \pm\int_{r_0}^{r} \frac{dr}{\rho\left(1 - \frac{r_g}{r}\right)\sqrt{f_L(r,\rho)}}, \qquad \varphi(r,\rho) - \varphi_0 = \pm\int_{r_0}^{r} \frac{dr}{r^2\sqrt{f_L(r,\rho)}}, \tag{4}$$

[1]Using the equations obtained in the next section, one can verify that this approximation is valid.

3.5. *(0.5 points)* Find $f_L(r,\rho)$.

3.6. *(0.5 points)* Draw a qualitative graph of the dependence $U(\rho, r^{-1}) = \rho^{-2} - f_L(r,\rho)$ from r^{-1} in the domain $r^{-1} \in [0, r_g^{-1}]$ for some fixed ρ. The graph should show all the minimum and maximum points of the function $U(\rho, r^{-1})$.

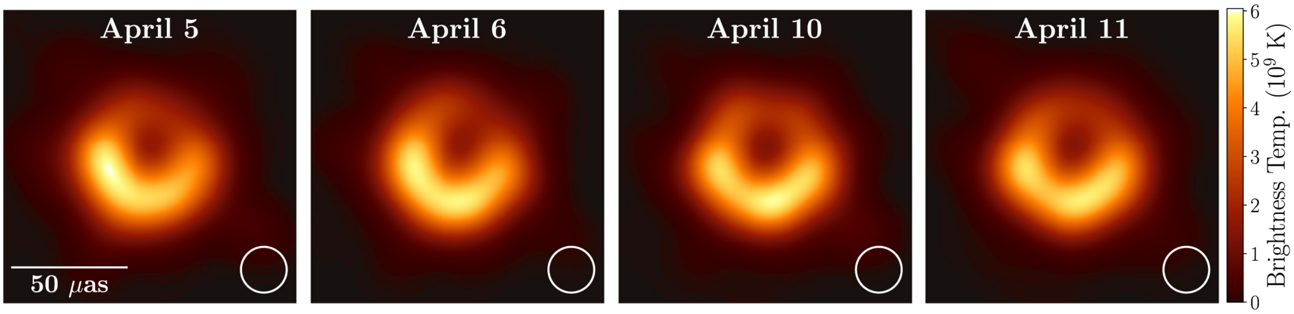
The region of allowed r is limited by the requirement that the the root expressions in (4) are non-negative. This means, in particular, that $U(\rho, r^{-1}) \leqslant \rho^{-2}$. From the graph constructed in item 3.6, it is clear that for different values of the parameter ρ, there are several types of light-like geodesics:

- For $\rho < \rho_{\min}$, the light ray falls into the black hole.
- For $\rho > \rho_{\min}$, there are two types of geodesics. One corresponds to a light ray coming from infinity, approaching the black hole to some minimum distance $r_{\min}(\rho)$, and then escaping back to infinity.

3.7. *(0.5 points)* Finf $\rho_{\min}/r_g$ accurate to 0,01.

3.8. *(0.5 points)* Find the minimum (among all $r_{\min}(\rho)$) distance $r_{\min}$ to which a light-like geodesic can approach the black hole without falling in, accurate to $0.01 r_g$.

In 2019, members of the Event Horizon Telescope collaboration published the image of the shadow of the supermassive black hole at the center of the galaxy $M87$:



Let us assume that the diameter of the bright ring in these images is equal to $2\rho_{\min}$. The diameter of the ring can conveniently be calculated as

$$d = \frac{d_{\text{out}} + d_{\text{in}}}{2},$$

where d_{out} and d_{in} are the inner and outer radii of the ring, respectively. Assume that the distance to the black hole is 16,4 million parsecs.

3.9. *(0.5 points)* Using the image above, find d_{out}. Express your answer in kilometers.

3.10. *(0.5 points)* Using the image above, find d_{in}. Express your answer in kilometers.

3.11. *(0 points)* Using the results obtained and the expression for $\rho_{\min}$ (see 3.7), find r_g for the supermassive black hole at the center of the galaxy M87. Express your answer in kilometers.

3.12. *(0 points)* Using the result from the previous item, find the mass of this black hole. Express your answer in solar masses.