Hint 2

IMPORTANT! The next task is both a hint and an alternative to the main task. Three important points:

1. You can continue to send the solution to the main problem.
2. At any moment before the final deadline you can start to solve the Alternative problem. If you do so, at the beginning of the solution write: *I am doing the Alternative problem!* In this case a penalty coefficient for the Alternative problem is

$$0{,}7 \cdot \sum_i \frac{k_i \cdot p_i}{10},$$

where p_i is a point for the problem item, and k_i is a penalty coefficient for the corresponding problem's item at the moment of moving to the Alternative problem. In other words, maximal points for the alternative problem equals to the maximal points you can gain at the moment of moving to the alternative one multiplied by 0,7. Also, we remind you that a penalty coefficient can't be less than 0,1. Solutions of the main problems from that moment will not be checked. Be careful!

3. The task consists of several items. The penalty multiplier earned **before** is applied to all points. In the future, each item is evaluated as a separate task. If you send a solution without any item, this item's solution is considered as Incorrect. For more information about scoring points for composite tasks, see the rules of the Cup. **Since switching to an alternative selection, there is no opportunity to return to solving the main task.** Also, after switching to an alternative task **the points for the main task are reset**.

Alternative task

It seemed as if this place had been created for Hidden. He took a shiny flask out of his travel bag, took a small sip, and felt a tart yet pleasant taste in his mouth. The evening breeze brushed against Hidden's face, while the silence seemed to settle over this City like a lulling dome.

Throwing back the hood of his newly purchased cloak, he walked along the deserted streets, regretting that there was no familiar crowd around him — one in which he could always easily disappear. Hidden knew exactly how much time had passed since his arrival in the City. He flipped open the lid of his pocket watch to make sure he would be precisely on time at the appointed place. The watch was the latest model, equipped with a built-in compass (as well as a barometer and a host of other instruments).

After checking his direction, he moved toward the avenue and saw a tall, beautiful building, its Ionic columns supporting a shingled roof adorned with an intriguing bas-relief at the top. Hidden approached the heavy door and, with some effort, pushed it open to enter.

Inside, it was surprisingly warm, though it seemed the stone walls should have given off a chill. Hidden walked to the center of the large hall, along the left wall of which stood tall bookcases reaching up to the ceiling, filled with books in green bindings decorated with gold embossing, while on the right hung beautiful tapestries depicting sleeping lions, grazing bulls, and birds soaring in the sky. Hidden smiled when he saw, deep in the room, a mirror with which he had always had an excellent relationship.

The heavy door behind him slammed shut with a resounding thud, breaking the silence of the sleeping City. Hidden turned around unhurriedly: before him stood a woman with a leonine bearing and an aquiline profile. Her unblinking eyes, devoid of eyelashes, stared piercingly at him. On her left arm dozed a hairless cat, which she absentmindedly stroked with slender fingers.

"You will stay here and never hide again. Answer correctly, and you will have the chance to leave," she said.

Part 1

Let's consider light propagation in a plane (x, y) in a medium with a refraction index $n(x)$. Optical path length for such refraction index is given by

$$s = \int_{x_A}^{x_B} n(x)\sqrt{1 + \left(\frac{dy(x)}{dx}\right)^2}\, dx. \tag{1}$$

A reminder, that the optical path length is defined for any function $y(x)$.

This optical path has two symmetries. By "symmetry" we imply a certain transformation of the function $y(x)$[1], which does not change s. Symmetries can be discrete or continuous. Continuous ones have continuous parameters. For example, if the system is symmetric with respect to rotation, the rotation angle serves as such parameter. Central symmetry can be an example of discrete symmetry.

[1]Examples of some function transformations $y(x)$: $y(x) \to (y(x))^2$, $y(x) \to y(-x) + 3x^3$. It can be seen that these transformation are not symmetries. (1).

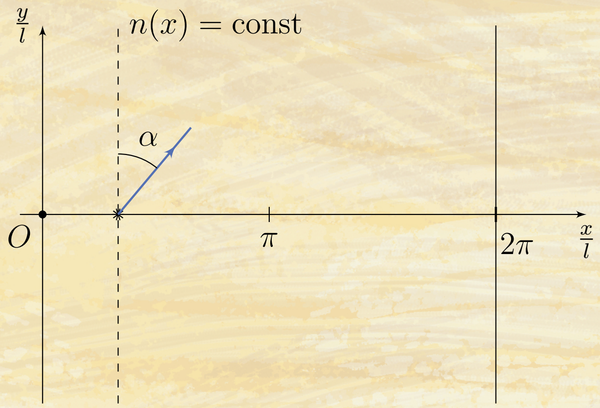
1. *(0 points)* Find a continuous symmetry of the optical path (1). What properties of the path result from this symmetry?

2. *(0 points)* Find a discrete symmetry of the optical path (1). Prove using this symmetry that path of a ray, emitted parallel to the x axis — is a straight line.

Now, let the refractive index take form

$$n(x) = n_0 \left(1 - \frac{1}{2}\cos\left(\frac{x}{l}\right)\right),$$

and the ray propagate in the region $x \in [0, 2\pi]$.



Let a point light source be located in a point S with coordinates $(\pi l/3{,}0)$ in a coordinate system (x, y). Let's consider a ray emitted from this point with an angle α to the y axis.

3. *(6 points)* Find the maximum value x/l for optical path of this ray.

Part 2

Medium is different now and refraction index is given by

$$n(x{,}y) = m(x)e^{-ky}.$$

We still consider the optical path of the ray emitted from point S with an angle α_S to the y axis. Let this path pass through a point Φ, where the angle between the tangent to the path and the y axis equals α_Φ.

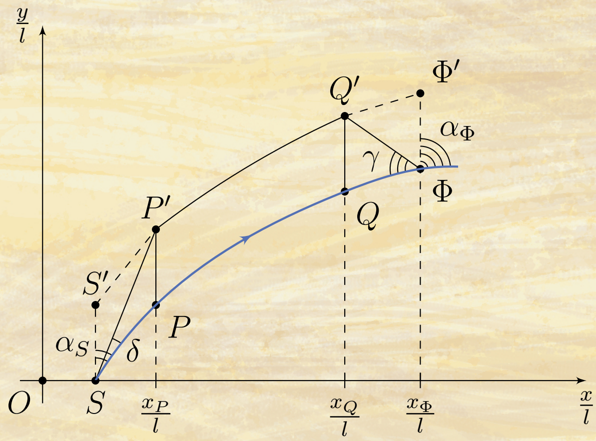
Let's consider curve $S'P'Q'\Phi'$, obtained by translating optical path segment $SPQ\Phi$ by λ, i. e. $SS' = PP' = QQ' = \Phi\Phi' = \lambda$ (see the figure below). Let us choose the parameter λ to be small compared to SP and $Q\Phi$. Then, the angles δ and γ will also be small. Let us choose points P and S such that the distances $|x_p - x_s|$ and $|x_Q - x_\Phi|$ are small compared to the optical path length.

1. *(0,5 points)* Find the optical path length $s_{SP'}$ in terms of optical path s_{SP}, angles δ, α_S, parameter λ, and function $n(x, y)$.

2. *(0,5 points)* Expand $s_{SP'}$ to the first order of smallness (see the Hint 0).

3. *(0,5 points)* Find the optical path length $s_{Q\Phi}$ in terms of optical path $s_{Q'\Phi}$, angles γ, α_Φ, parameter λ, and function $n(x, y)$.

4. *(0,5 points)* Expand $s_{Q\Phi}$ to the first order of smallness (see the Hint 0).



5. *(1 point)* Find the optical path length $s_{P'Q'}$ in terms of optical path s_{PQ}, parameters λ and k.

6. *(0,5 points)* Expand the equation for $s_{P'Q'}$ to the first order of smallness λ.

7. *(0,5 points)* Using the equality of optical path lengths $s_{SPQ\Phi}$, $s_{SP'Q'\Phi}$ in the first order of smallness get an equation for s_{PQ} in terms of $n(x,y)$, k, α_S and α_Φ.

Since P and Q are close to S and Φ, optical path s_{PQ} is equal to $s_{SPQ\Phi}$ in the zeroth order.

Part 3

Let us consider a medium with a refractive index

$$n(x,y) = \frac{x}{(x^2+y^2)^2}. \tag{2}$$

Optical path length for such medium is given by

$$s = \int \frac{x}{(x^2+y^2)^2}\sqrt{dx^2+dy^2}.$$

One can see that there exists a coordinate transformation $x'(x,y)$, $y'(x,y)$, in which this optical path takes the same form as (1):

$$x'(x,y) = \frac{x}{x^2+y^2}, \quad y'(x,y) = \frac{y}{x^2+y^2}.$$

It can be shown that the coordinate transformation $(x,y) \to (x',y')$ preserves the angles. In order to show that, let's consider a small triangle in (x,y) plane. Under such a transformation, to first order of smallness, it will map into a small triangle in the (x',y') plane. In order to show that this transformation preserves the angles let's prove that these triangles are similar. For that we will find ratio of the lengths of their sides. Length of any small segment in (x,y) plane is equal to $dl_{(x,y)} = \sqrt{dx^2+dy^2}$, and in (x',y') plane its equal to [2] $dl_{(x',y')} = \sqrt{(dx'(x,y))^2 + (dy'(x,y))^2}$. It can be shown that $dl_{(x',y')} = f(x,y)dl_{x,y}$. From this relation, it follows that the triangles are similar and that angles are preserved.

1. *(0 points)* Find $f(x,y)$.

[2] $dx'(x,y)$ means $(x'(x+dx, y+dy) - x'(x,y))_1$ in the Hint 0 notation.

In such coordinates the optical path length is given by

$$s = \int m(x') \sqrt{dx'^2 + dy'^2}.$$

2. *(0 points)* Find $m(x')$.

This form of optical path length indicates that the problem of wave propagation in a medium with a refractive index given by (2) is equivalent to the problem of wave propagation in a stack of plates with a refractive index $m(x')$.