LPR Cup

11.s05.e02

I wash my hands of this weirdness
Pirates of the Caribbean: At World's End

Parte Uno. El Cuadrado

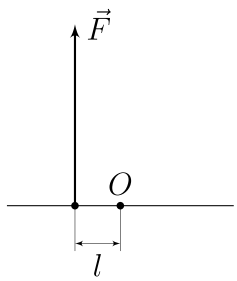
Listened to the advice of Ant-Traveler, Hedgehog decided to go on a cruise from Cuba to Aruba. The gentlemen who undertook to bring Hedgehog accepted payment only ~~in slaves: Romeo and Ringo, rum, roman sculptures, earrings~~ in square coins. Hedgehog managed to get out of the situation, pay the fare, and was given a bizarre coin in the form of Small Square with a side of L and a mass of m. On the two sides of the Small Square were minted LPR Cup and LPR Cube.

Hedgehog thought the coin was fake, and the gentlemen who gave it to him looked suspicious, so Hedgehog decided to throw it into the sea. To his surprise, some time later the coin was again in his pocket. This intrigued Hedgehog, and he decided to study the Small Square in more detail.

The first thing Hedgehog noticed is that Small Square is thin and uniform. Then he proceeded to conduct the following experiment with it.

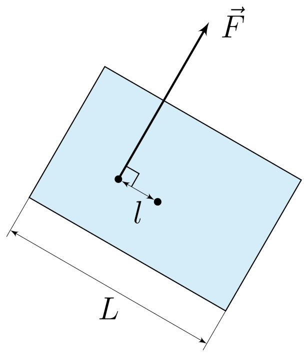
Hedgehog throws horizontally positioned Small Square from a height of H as follows: Hedgehog applies a short impulse of force to the Small Square at a random distance $l \in \left[0, \frac{L}{2}\right]$ from its center in such a way that the final velocity of the Small Square's center of mass equals to v_0 and is directed vertically upwards.

Hedgehog decided to give you a Hint 0: the Small Square is subjected to some force F for a short time τ. For a solid body, equations of motion can be written as



$$\begin{cases} F = ma_c, \\ M = I\varepsilon, \end{cases}$$

where F is the force applied by Hedgehog, a_c is the acceleration of the center of mass, I is the moment of inertia relative about an axis passing through the center of mass, ε is the angular acceleration, M is the torque of force F about an axis passing through the center of mass. Using the fact that $M = Fl$, let's reduce terms:



$$I\varepsilon = lma_c.$$

Since $\varepsilon = \frac{\Delta\omega}{\Delta t}$, and $a_c = \frac{\Delta v}{\Delta t}$, we obtain the angular velocity of Small Square:

$$\Delta\omega = \frac{ml\Delta v}{I} \Rightarrow \omega = \frac{mlv_0}{I}.$$

We obtained the last equality by summing small increments. For Small Square consider the angular momentum to be known and equal to $I = \frac{1}{12}mL^2$.

Initially from above Small Square minted Cube. Consider that when touching the deck Small Square loses all its linear and angular velocity.

Find the probability of the LPR Cup falling out, if all possible values of $l \in \left[0, \frac{L}{2}\right]$ are equally probable, and the system parameters are:

1. *(2 points)* $L = 23{,}00$ mm, $m = 5{,}630$ g, $v_0 = 0{,}8800$ m/s, $H = 5{,}000$ cm, $g = 10{,}00$ m/s^2
2. *(3 points)* $L = 23{,}00$ mm, $m = 5{,}630$ g, $v_0 = 4{,}500$ m/s, $H = 100{,}00$ cm, $g = 10{,}00$ m/s^2

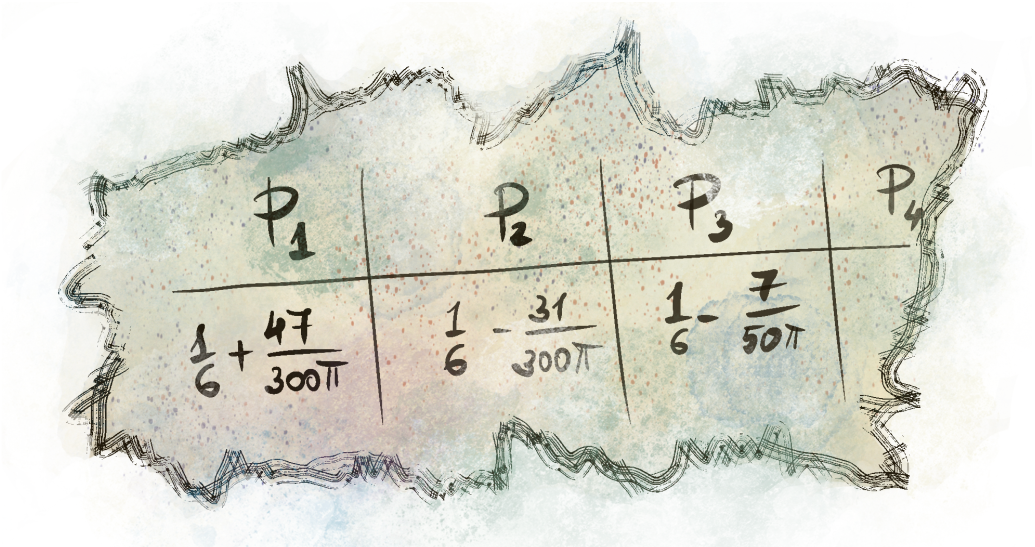
Give the answer with no less than 4 significant figures. Neglect air resistance.

Parte Dos. El Cubo-pícaro

When Hedgehog was playing dice with the gentlemen in the cabin at night, one of the dice seemed suspicious to him. Seizing the opportunity, Hedgehog discreetly requisitioned Cube to study it in more detail. To avoid being noticed, Hedgehog climbed into a crow's nest and decided to learn the probability distribution[1] of the Cube's faces.

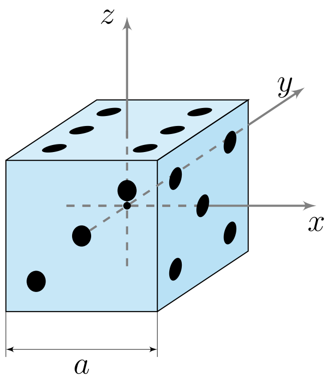
Hedgehog places Cube on a random edge at a random angle to the horizon, so that all angles and edges are equally probable. Hedgehog releases Cube, and it falls onto one of the faces. Consider the impact to be perfectly inelastic. After this, Hedgehog records the result in his diary. As dawn breaks, Hedgehog places Cube on the edges a sufficiently large number of times. He recalculates the probability table and confirms that his suspicions were not unfounded.

After being satisfied with the work done, Hedgehog decided to go rest in his cabin. While he was descending from the crow's nest, a knowledge-hungry seagull attacked him. There was a ~~BLOODY~~ battle for the diary with the notes. Hedgehog has paws, so all he managed to save was a small scrap of paper, on which only half of the table was written. Help Hedgehog restore all the probabilities, if the remaining records look like this:



[1] The probability distribution is a law that describes the range of values of a random variable and the corresponding probabilities of these values occurring.

1. *(1 point)* Let the center of mass be weakly shifted by $\delta\vec{r} = (x,\, y,\, z)$ from the center of the Cube (see figure). Express the probabilities of the Cube's faces falling in terms of the displacement of the center of mass. Simplify the obtained expressions using approximate formulas. Take into account small terms of $\frac{x}{a}$, $\frac{y}{a}$, $\frac{z}{a}$ up to the second order of smallness inclusive.



2. *(2 points)* Find the sums of probabilities for opposite faces. Take into account small terms of $\frac{x}{a}$, $\frac{y}{a}$, $\frac{z}{a}$ up to the third order of smallness inclusive.
3. *(2 points)* Help Hedgehog restore the probabilities of each face falling, using the data that has been preserved in the table. Take into account small terms of $\frac{x}{a}$, $\frac{y}{a}$, $\frac{z}{a}$ up to the second order of smallness inclusive.

Note 1: the sum of the numbers on opposite faces is 7.
Note 2: Stay calm, the sea state is calm. Therefore, the shaking of the deck can be neglected.

Since Hedgehog was an astronomer and was passionate about star maps, he knew several amusing and useful facts that might come in handy for you:

$$\frac{1}{1+\xi} \approx 1 - \xi + \xi^2 - \xi^3,\ |\xi| \ll 1$$

$$\sqrt{1+\xi} \approx 1 + \frac{\xi}{2} - \frac{\xi^2}{8} + \frac{\xi^3}{16},\ |\xi| \ll 1$$

$$\arcsin\left(\Xi_0 + \xi\right) \approx \arcsin\left(\Xi_0\right) + \frac{\xi}{\sqrt{1-\Xi_0^2}} + \frac{\xi^2 \Xi_0}{2\left(1-\Xi_0^2\right)^{3/2}} + \frac{\xi^3\left(2\Xi_0^2+1\right)}{6\left(1-\Xi_0^2\right)^{5/2}},\ |\xi| \ll 1$$

$$\arccos\left(\Xi_0 + \xi\right) \approx \arccos\left(\Xi_0\right) - \frac{\xi}{\sqrt{1-\Xi_0^2}} - \frac{\xi^2 \Xi_0}{2\left(1-\Xi_0^2\right)^{3/2}} - \frac{\xi^3\left(2\Xi_0^2+1\right)}{6\left(1-\Xi_0^2\right)^{5/2}},\ |\xi| \ll 1$$

$$\operatorname{arctg}\left(\Xi_0 + \xi\right) \approx \operatorname{arctg}\left(\Xi_0\right) + \frac{\xi}{\Xi_0^2+1} - \frac{\xi^2 \Xi_0}{\left(\Xi_0^2+1\right)^2} + \frac{\xi^3\left(3\Xi_0^2-1\right)}{3\left(\Xi_0^2+1\right)^3},\ |\xi| \ll 1$$

$$\operatorname{arcctg}\left(\Xi_0 + \xi\right) \approx \operatorname{arcctg}\left(\Xi_0\right) - \frac{\xi}{\Xi_0^2+1} + \frac{\xi^2 \Xi_0}{\left(\Xi_0^2+1\right)^2} + \frac{\xi^3\left(1-3\Xi_0^2\right)}{3\left(\Xi_0^2+1\right)^3},\ |\xi| \ll 1$$

$$\sin\left(\Xi_0 + \xi\right) \approx \sin\left(\Xi_0\right) + \xi\cos\left(\Xi_0\right) - \frac{1}{2}\xi^2\sin\left(\Xi_0\right) - \frac{1}{6}\xi^3\cos\left(\Xi_0\right),\ |\xi| \ll 1$$

$$\cos\left(\Xi_0 + \xi\right) \approx \cos\left(\Xi_0\right) - \xi\sin\left(\Xi_0\right) - \frac{1}{2}\xi^2\cos\left(\Xi_0\right) + \frac{1}{6}\xi^3\sin\left(\Xi_0\right),\ |\xi| \ll 1\ (1+\xi)^n \approx 1 + n\xi + \frac{1}{2}n(n-1)\xi^2$$

An example from the notes in the margins:

$$\frac{1}{\Xi_0+\xi+\zeta} = \frac{1}{\Xi_0}\cdot\frac{1}{1+\xi/\Xi_0+\zeta/\Xi_0} \approx \frac{1}{\Xi_0}\left(1-\underbrace{(\xi/\Xi_0+\zeta/\Xi_0)}_{\text{First order}}+\underbrace{\left((\xi/\Xi_0)^2+2\frac{\zeta\xi}{\Xi_0^2}+(\zeta/\Xi_0)^2\right)}_{\text{Second order}}\right)$$

We insistently ~~insist on~~ recommend using mathematical packages ~~of goodies and yummies~~.

First hint — 06.05.2024 20:00 (Moscow time)
Second hint — 08.05.2024 12:00 (Moscow time)

Final of the second round — 10.05.2024 20:00 (Moscow time)