The easiest way to avoid traps is the one who knows how to set them.
Publilius

Problems-Cubies

If you have any questions regarding mathematics you should text Philipp. For questions about programming text Dima

Part 1. (3 points) Single-qubit gates in a quantum computer

A quantum bit or qubit is a unit of memory storage in a quantum computer. The qubit differs from its younger brother and its classical counterpart, the bit, which takes only two values — 0 and 1, in that it is a quantum mechanical system with quantum properties. These properties are taken into account and the description of the work of qubits is carried out using a properly selected mathematical apparatus. In this part of the task, we will get acquainted with the cubies themselves and the basic operations with them.

Qubit state

In atoms, the energy of an electron often cannot change continuously, but takes discrete values (for example, in a hydrogen atom). A qubit is a model of a quantum mechanical system in which only two energy levels are considered. For example, in one of the computers in the RQC, a calcium ion is used as qubit, one of the levels of which is the ground state of an external electron, and the other is one of its excited states.

The state of the qubit is described by a two-dimensional complex vector

$$\psi = \begin{pmatrix} c_1 \\ c_0 \end{pmatrix},$$

where c_0 and c_1 are complex functions called probability amplitudes for which the condition $|c_0|^2 + |c_1|^2 = 1$ is satisfied.

If $\psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then this is the ground state, similar to when the classical bit takes the value 0.

If $\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then this is an excited state, similar to when the classical bit takes the value 1.

An arbitrary state of the qubit can be represented in the form of:

$$\psi = c_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_0 \psi_0 + c_1 \psi_1,$$

We can also say that the functions c_0 and c_1 are the amplitudes of the detection of the qubit in the states c_0 and c_1. It follows from quantum mechanics that if we make any measurement of the state of a qubit, then it will necessarily turn out to be either in the state ψ_0 or in the state ψ_1. The number $|c_0|^2$ shows the probability that after measuring the state of the qubit it will be in the state 0, and $|c_1|^2$ – the probability that it will be in the state 1. Before measuring the qubit is in the state $\psi = \begin{pmatrix} c_1 \\ c_0 \end{pmatrix}$, i.e. in a superposition of these two states. [1]

Example 1. The qubit is in the state $\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$. If we measure the state of a quantum mechanical system, then it will be in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with probability 1/4, and in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with probability 3/4.

Example 2. The qubit is in the state $\begin{pmatrix} i/\sqrt{2} \\ \sqrt{2}/2 \end{pmatrix}$. The probabilities that after measuring the state of the qubit, it will be in the state of ψ_0 or ψ_1 are equal to 1/2.

1.1. *(0.1 points)*. It is known that the qubit is in the state $\begin{pmatrix} (1+i)/2 \\ x \end{pmatrix}$. Find x.

From the fact that $|c_0|^2 + |c_1|^2 = 1$, it follows that one of the ways in which you can set a general view of the qubit state is the expression:

$$\psi = \cos\left(\frac{\theta}{2}\right) \psi_0 + e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \psi_1.$$

Such parametrization is often chosen because of its convenient and visual interpretation through the Bloch sphere (see Fig. 1). On it, the state is characterized by a unit vector forming an angle θ with the z-axis. It's projection on the xy plane forms an angle φ with the x-axis.

It's important to note that in general the state is defined up to the global phase:

$$\psi = e^{i\alpha}\left(\cos\left(\frac{\theta}{2}\right) \psi_0 + e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \psi_1\right).$$

Actions with "cubies"

Similar to logical operations in a classical computer (AND, OR, NOT, etc.), in a quantum computer, calculations are performed using quantum operations on qubits, which are also called quantum gates. There are single-qubit gates that act only on one qubit, and multi-qubit gates

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[1]A quantum mechanical system differs from a classical one in that when it is measured, its state changes fundamentally. In the case of qubits, after the measurement, it turns out to be either in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
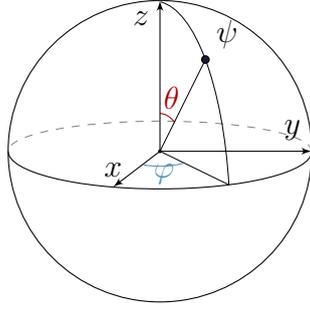
Рис. 1: The Bloch sphere

that act on a set of cubes of size 2 or more. The gate action changes the state of the qubit and can be represented by a matrix acting on the vector ψ. In this part of the task, we suggest you work with single-bit gates.

The matrix of a one-bit gate is a complex-valued matrix of size 2×2:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

with the following properties: $U^\dagger U = U U^\dagger = I$, where $U^\dagger = \left(U^T\right)^*$ is the transposition and complex conjugation of the matrix U, I is the unit matrix. Matrices having this property are also called unitary matrices.

The effect of the matrix on the qubit state is described by the usual matrix multiplication of the matrix by a column vector:

$$U\psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} c_1 \\ c_0 \end{pmatrix}.$$

1.2. *(0.1 point)* Calculate the probability with which the qubit will be in state 0 after the action of gate U on it, considering the elements A, B, C and D and c_1, c_0 known.

1.3. *(0.1 point)* Arbitrary linear transformation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ it is defined by eight real parameters. The requirement of unitarity reduces the number of independent parameters. How many independent real parameters uniquely defines the unitary transformation.

1.4. *(0.1 point)* Find the unitary transformation U if it is known that $U^2 = U$.

1.5. *(0.1 point)* Prove that sequentially executed one-qubit gates are also one-qubit gates.

Single-cube gates are often laid out through Pauli matrices, which have the following form:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1.6. *(0 point)* Check that the Pauli matrices are unitary.

1.7. *(0.2 point)* Act by each Pauli matrix on the vector $\begin{pmatrix} i/\sqrt{2} \\ \sqrt{2}/2 \end{pmatrix}$. How will the amplitudes of the detection of the states ψ_0 and ψ_1 change. Draw the initial and received states on the Bloch sphere.

1.8. *(0.1 point)* What geometric meaning does each of the transformations have?

1.9. *(0.1 point)* Find the difference $XY - YX$.

We introduce the Hadamard element (gate):

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

1.10. *(0.1 points)* Find what the following matrices are equal to:

$$HXH, \quad HYH, \quad HZH.$$

1.11. *(0.1 point)* Act with the Hadamard element on the states $(\psi_0 + \psi_1)/\sqrt{2}$. What will be the probabilities of detecting a qubit in the states ψ_0 and ψ_1 after the date of this gate. How will the answer change if the initial state was $(\psi_1 - \psi_0)/\sqrt{2}$.

When solving this point, the following formulas may be useful to you. Let 's define the concept of the exponent of the matrix:

$$e^{\alpha A} + I + \alpha A + \frac{\alpha^2}{2!} A^2 + \ldots + \frac{\alpha^n}{n!} A^n + \ldots$$

1.12. *(0.4 points)* Find $R_x(\theta) = \exp(-i\theta X/2), R_y(\theta) = \exp(-i\theta Y/2), R_z(\theta) = \exp(-i\theta Z/2)$. Consider θ to be a real number.

1.13. *(0.1 points)* Are the R_x, R_y, R_z one-bit gates?

1.14. *(0.3 points)* What is the geometric meaning of each of the transformations?

It follows from the previous paragraphs and their geometric interpretation that an arbitrary one-bit gate can be represented as follows:

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta),$$

where α, β, γ and δ are some real numbers.

1.15. *(0.2 points)* Select the parameters α, β, γ and δ to get the Hadamard gate.

1.16. Offer a one-bit gate to move from the state $c_0 = 1/\sqrt{2}$ and $c_1 = 1/\sqrt{2}$

    (a) *(0.1 points)* in $c_0 = \sqrt{2}/2$ and $c_1 = i\sqrt{2})/2$,

    (b) *(0.1 points)* in $c_0 = \sqrt{3}/2$ and $c_1 = 1/2$,

    (c) *(0.1 points)* in $c_0 = \sqrt{3}/2$ and $c_1 = i/2$.

Rabi Oscillations

In an ion-powered quantum computer, any manipulation of cubes is performed using lasers. So, to perform single-qubit operations, it is necessary to shine a laser on the ion with a frequency exactly equal to the transition frequency. For calcium, this frequency is 411.0421297763931 THz (the wavelength of the laser at this frequency is 729.3472767939562 nm). The gate matrix in this case depends on the time during which we shine a laser on the ion, and is given by the following formula:

$$U(t) = \begin{pmatrix} \cos\left(\frac{\Omega_l t}{2}\right) & e^{-i\varphi + i\pi/2} \sin\left(\frac{\Omega_l t}{2}\right) \\ e^{i\varphi - i\pi/2} \sin\left(\frac{\Omega_l t}{2}\right) & \cos\left(\frac{\Omega_l t}{2}\right) \end{pmatrix},$$

Where Ω is a parameter characterizing the interaction of an ion with a laser, which is called the Rabi frequency, φ is the laser phase and t is the time. The Rabi frequency of ion-laser interaction is determined by the internal properties of the ion and the intensity of the laser beam.

1.17. *(0.1 points)* Find out what the gate time (the duration of the laser pulse) and the laser phase should be so that the gate matrix is equal to the Pauli matrix Y if the Rabi frequency is $1.6/2\pi$ MHz.

Fidelity of operations

In the laser, there are intensity and frequency noises and other imperfections of the experimental setup that lead to errors, as a result of which the real gate may differ from the one we wanted to implement. This leads to a loss of accuracy (fidelity) of operations.

The following rule is used to calculate the accuracy of operations. Let the initial qubit be in a certain state ψ, and we need to conduct a gate G, as a result of which the state $G\psi = g_0\psi_0 + g_1\psi_1$ will be obtained. During the experiment, another gate G was implemented, slightly different from the original one, and the state $G\psi = (g_0)\psi_0 + (g_1)\psi_1$ was obtained. Then fidelity F is calculated by the following formula:

$$F = |\tilde{g}_0^* g_0 + \tilde{g}_1^* g_1|^2.$$

Let's say our goal was to hold the Hadamard gate, but it turned out as always the transformation

$$\tilde{H} = \frac{1}{\sqrt{1 + 0.99^2}} \begin{pmatrix} 0.99 & 1 \\ 1 & -0.99 \end{pmatrix}.$$

1.18. *(0.1 points)* Find fidelity if the initial state was $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

1.19. *(0.1 points)* What will fidelity for the initial state from the previous point be equal to if we wanted to perform the Hadamard transformation 10 times and each time we received the matrix $\tilde{H}$?

1.20. *(0.1 points)* How will the answer to the question 20 of this paragraph change if the initial state was $\begin{pmatrix} i/\sqrt{2} \\ \sqrt{2}/2 \end{pmatrix}$.

Error correction

As for a classical computer, special codes have been developed for a quantum computer that allow correcting (correcting) errors that occur. In this paragraph, we will consider the classic method of error correction in order to understand the idea of the algorithm.

Let us consider one logical bit of a computer (the state of which, for example, we want to transmit over some distance) three physical bits, i.e. three ions in the same initial state. During the transfer of the state of a logical bit, one or more of the three physical bits can change their states. We will consider as an error only the situation in which a physical bit is flipped, i.e. its state is reversed, and let the probability of a flip of one physical bit be p.

After the transfer, the physical bits can either be in the correct state, or flip over and lead to an error. We will assume that the state of the logical bit will be the state in which most of

the physical bits turned out to be. That is, if after measuring we fix the result as 001, then we assume that the logical bit is in the state 0. And if the result is 101, then we assume that the state of the logical bit is 1. If two bits turned over at the same time after performing the operation, then we will determine the state by mistake. This scheme is the simplest error correction scheme, for which we increase the number of physical carriers of the information storage unit, and thereby increase the resistance to errors.

Before the transmission all the bits are in the same state.

1.21. *(0.1 points)* Find out what the probability of a flip of one physical bit p should be so that encoding with three bits is more reliable than with one.

1.22. *(0.1 points)* Let the probability of a flip of one physical bit be $p = 0.1$. How many such physical bits should be taken to encode one logical bit, so that in the presence of a classical error correction scheme, the probability of an incorrect reading of the state of a logical bit is less than 0.01. Assume that the number of bits is odd.

Part 2. (5 points) Paul's Trap

To create qubits, we need to localize individual ions in space. To convert calcium atoms into qubits, they are first separated from a piece of calcium by laser ablation, then ionized once, while the atom turns into a positively charged ion, and then captured in a Paul trap placed in an ultra-high vacuum. Further, deeper cooling cooling of ions to almost zero temperature allows to create the ground state of the qubit.

Capture and retention of ions in Paul traps

Introduction

2.1. Let the potential function in the neighborhood of the origin of the Cartesian coordinate system have the form:
$$\varphi(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2,$$

Where α, β and γ are some constants.

(a) *(0.2 points)* Find the projections of the electric field strength $E_x(x, y, z), E_y(x, y, z)$ and $E_z(x, y, z)$.

(b) *(0.3 points)* Find what the charge density is at the origin.

(c) *(0.1 points)* In which case it will be zero.

According to the well-known Irnshaw theorem, if there are only electrostatic fields, then the charge cannot be in a stable equilibrium position. However, if the electric fields are variable, then it is possible to create a situation where the trajectory of the charged particle will always be limited in space. To do this, it is necessary to choose the correct configuration of the electrodes, as well as the frequencies and amplitudes of the voltages applied to them. One of the examples where this was implemented is the Paul trap, the properties of which are proposed to be studied in this part of the problem.

A Paul trap is a set of electrodes that are supplied with constant and oscillating voltages at a radio frequency. With properly selected parameters of potentials and frequencies, charged particles can be captured in the space between the electrodes. The configuration of the electric fields of the Paul trap is described by the following analytical potential:

$$\Phi = \left(-\frac{V_{RF} \cos \Omega t}{r_0^2} - \frac{k_z U_{DC}}{z_0^2} \right) \frac{x^2}{2} - \left(-\frac{V_{RF} \cos \Omega t}{r_0^2} + \frac{k_z U_{DC}}{z_0^2} \right) \frac{y^2}{2} + \frac{k_z U_{DC}}{z_0^2} z^2, \qquad (1)$$

where Ω is the angular radio frequency of the electrodes, V_{RF} — voltages on radial electrodes of variable fields, r_0 and z_0 — geometric parameters of the trap (distances from the center of the trap to the corresponding electrodes), U_{DC} is the voltage at the axial electrodes, k_z is the geometric parameter of axial retention.

In our notation, the OX and OY axes are called the radial axes of the trap, and the OZ axis is called the axial axis.

2.2. *(0.1 points)* Assume $V_{RF} = 0$. Draw the trajectory of a particle, if at the beginning it is located at point with coordinates $(1; 0; 0)$ and its initial velocity equals zero.

2.3. *(0.1 points)* Consider plane $z = 0$. Let $V_{RF}/r_0^2 = 2k_z U_{DC}/z_0^2$. Draw the dependence $\Phi(x, z, 0)$ at the $t = 0$ and in the moment $t = \pi/\Omega$.

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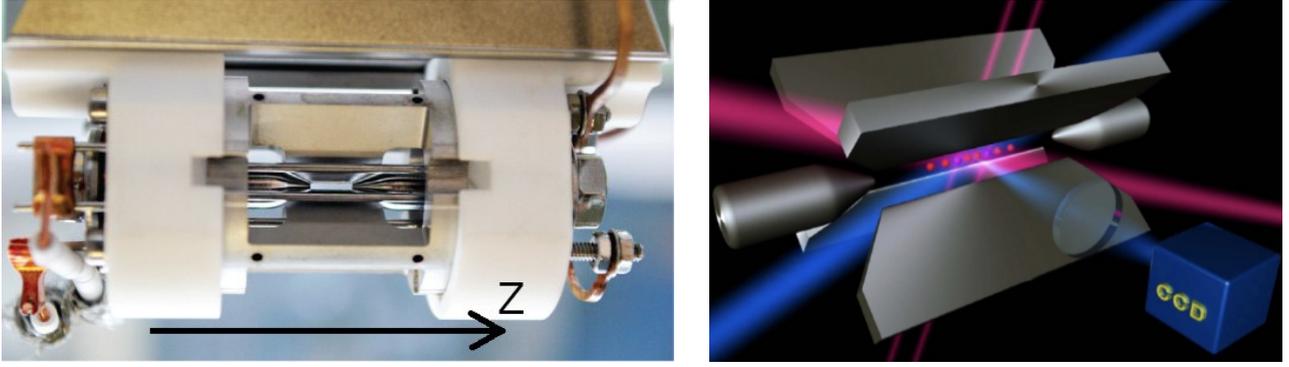
Рис. 2: Photo and diagram of Paul's trap, from the quantum website optics.at Rainer Blatt Laboratories, Innsbruck, Austria

The equations of motion for a particle of mass m and charge e in a given potential, following from (1), are called Mathieu equations and have the form:

$$\begin{cases} \dfrac{d^2x}{d\tau^2} + (a_x - 2q_x \cos 2\tau)\, x = 0, \\ \dfrac{d^2y}{d\tau^2} + (a_y - 2q_y \cos 2\tau)\, y = 0, \\ \dfrac{d^2z}{d^2\tau} + a_z z = 0, \end{cases} \tag{2}$$

where the parameters of the equations are described by the following formulas:

$$\tau = \frac{\Omega}{2} t, \quad a_x = a_y = -\frac{a_z}{2} = -\frac{4 e k_z U_{DC}}{m z_0^2 \Omega^2}, \quad q_x = -q_y = q = \frac{2 e V_{RF}}{m r_0^2 \Omega^2}.$$

Note that the parameters of the Mathieu equations depend on the experimental characteristics of the trap: electrode voltages, radio frequency and geometric parameters. Consequently, the properties of the Mathieu equations (and, as a consequence, their solutions) can be controlled by a corresponding change in the parameters of the Paul trap.

If the trap parameters are well chosen, the solutions of the Mathieu equations (2) are such that the ion trajectory is limited in space. We will assume that it is captured if its trajectory is limited and the amplitude of the ion oscillations is less than the geometric dimensions of the trap (less than 1 mm). For such a capture, the values of parameters a and q in all directions of motion (axial and radial) must belong to the **stability zone**. An example of a stability diagram for some Paul trap is shown in the figure. The shaded area is the stability zone, the unshaded area is the parameter area at which the ions leave the trap.

Numerical simulation

In this part of the problem, we suggest that you numerically simulate the equations of the trajectory of ions in a Paul trap and study their properties.

So that you don't have to program anything, we have prepared notebook in the Google Colab environment. It can be used without SMS, but with registration. Go to notebook link (just in case, here it is again). go to your Google account, click on the code execution and you are happy. Yes, cells can be hidden and opened by clicking on the triangle. It has become more
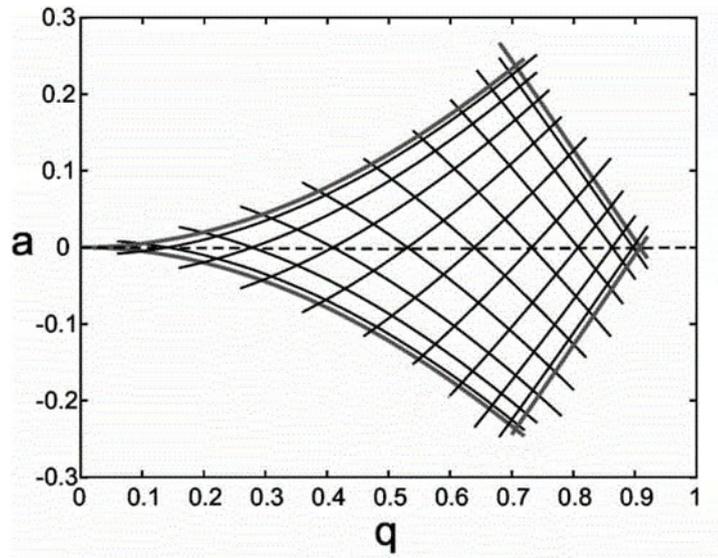
convenient to work! Details about how to use the notebook are described in this video. ~~If~~ When you ~~remain~~ have questions about how to use this notebook, then you can ask them in private messages Dima. Do not hesitate, he is ready and charged to answer 25 (twenty-five) to 7.

Remark. In the notebook the parameter RF_frequency is not the angular frequency, it is connected to the parameter in equation 1 with a formula RF_frequency $= \Omega/2\pi$.

2.4. *(0.3 points)* Let $a_x = -0.1$. At what values of the parameter q_x in the range from -2 to 2, the trajectories of ions along the x-axis will be limited in space. As an answer, give the intervals of the q_x parameter. Determine the boundaries of the interval with an accuracy of 0.1. Will it also be trapped?

2.5. *(0.3 points)* Let $a_x = 0.1$. At what values of the parameter q_x in the range from -2 to 2, the trajectories of ions along the x axis will be limited in space. As an answer, give the intervals of the q_x parameter. Determine the boundaries of the interval with an accuracy of 0.1. Will it also be trapped?

2.6. *(0.3 points)* Let $a_y = -0.01$. At what values of the parameter q_y in the range from -2 to 2, the trajectories of ions along the x axis will be limited in space. As an answer, give the intervals of the parameter q_y. Determine the boundaries of the interval with an accuracy of 0.1. Will it also be trapped?

Doppler cooling of ions in a Paul trap

For more efficient operation of the Paul trap, the ions are cooled using the Doppler effect. In this part of the task, we will study in more detail how this is done.

2.7. *(0 points)* The source emits light at a frequency of ω_0 and moves along a straight line connecting it and the receiver at a speed of v_0. What is the radiation frequency that the receiver will detect?

2.8. *(0.2 points)* The source and receiver are located at points A and B, respectively. The source moves at a velocity v_0 perpendicular to AB, the receiver moves at a velocity v_0 along the line AB (see figure). What is the frequency of radiation that the receiver will detect if it emits a signal at a frequency of ω_0 in the reference frame of the source?

2.9. *(0.1 points)* Will consider the ideal gas. The probability that the velocity of a gas atom lies in the range from v to $v + dv$ is determined by the expression $N \cdot 4\pi v^2 e^{-mv^2/2kT} dv$, where m is the mass of the atom, k is the Boltzmann constant, T is the temperature of the gas, N is the normalization constant multiplier. Find the most probable speed of the atom. The most probable velocity is the velocity close to which the largest number of molecules has.

2.10. *(0 points)* The atom moves at a speed of v_0 towards the laser beam. The electron is in the ground state. An electron can go into an excited state if it absorbs a photon with a frequency of ω_0. What should be the detuning of the laser $\Delta\omega$ from the frequency ω_0 so that the atom absorbs the photon?

2.11. *(0 points)* The pulse of a photon with a frequency of ω is determined by the expression $\hbar\omega/c$. Find the speed of the atom from the previous point, if it absorbed a photon.

2.12. *(0.3 points)* In the experimental setup, a stream of single sodium atoms flies out at an average speed of 10^5 cm/s towards a laser source with a wavelength of 363.8 nm. When interacting with laser radiation, the atom absorbs a photon and passes from the main energy state to the excited one. With further movement, the atom can return to its original state with the forced or spontaneous emission of a photon. The forced emission of a photon is provoked by interaction with an external photon and the emitted photon has the same energy value and direction of propagation as the provoking one. And when a new photon is spontaneously emitted, it has the same energy value as the photon absorbed earlier, but in a random direction. Considering the processes of forced and spontaneous radiation equally probable, calculate the average number of absorption of photons of laser radiation by sodium atoms required to "cool"the atoms to zero velocity.

Doppler cooling is often used in experiments with Paul traps. In the following paragraphs, the situation without Doppler cooling will be analyzed first, and in point 15 we will implement it.

If the ions are sufficiently cooled (to a temperature[2] less than mK), the approximate solution along the radial coordinates is described by the following formula:

$$x(t) \propto \cos\left(\frac{\omega_x t}{2}\right)\left[1 - \frac{q_x}{2}\cos\Omega t\right],$$

where the first multiplier is oscillations at the secular frequency $\omega_x/2$, and the second multiplier describes small and frequent ion oscillations away from the center of the trap.

2.13. *(0.1 points)* Select the parameters Ω, U_{DC} and V_{RF} so as to capture the ion. Physical voltages are from 1 to 1000 V. The radio frequency can be changed from 20 to 30 MHz. Get pictures in such a way that the oscillations at the secular frequency and micro-movements are necessarily visible. You should get pictures of something like the following. As a response, send parameter values and graphs with signed axes and units of measurement.

2.14. *(0.2 points)* From the graphs of coordinates from time, find the secular oscillation frequencies of the ion in the trap. Compare with the frequencies obtained in the spectrum. Check

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[2]the temperature of an ion is determined by its kinetic energy divided by the Boltzmann constant k
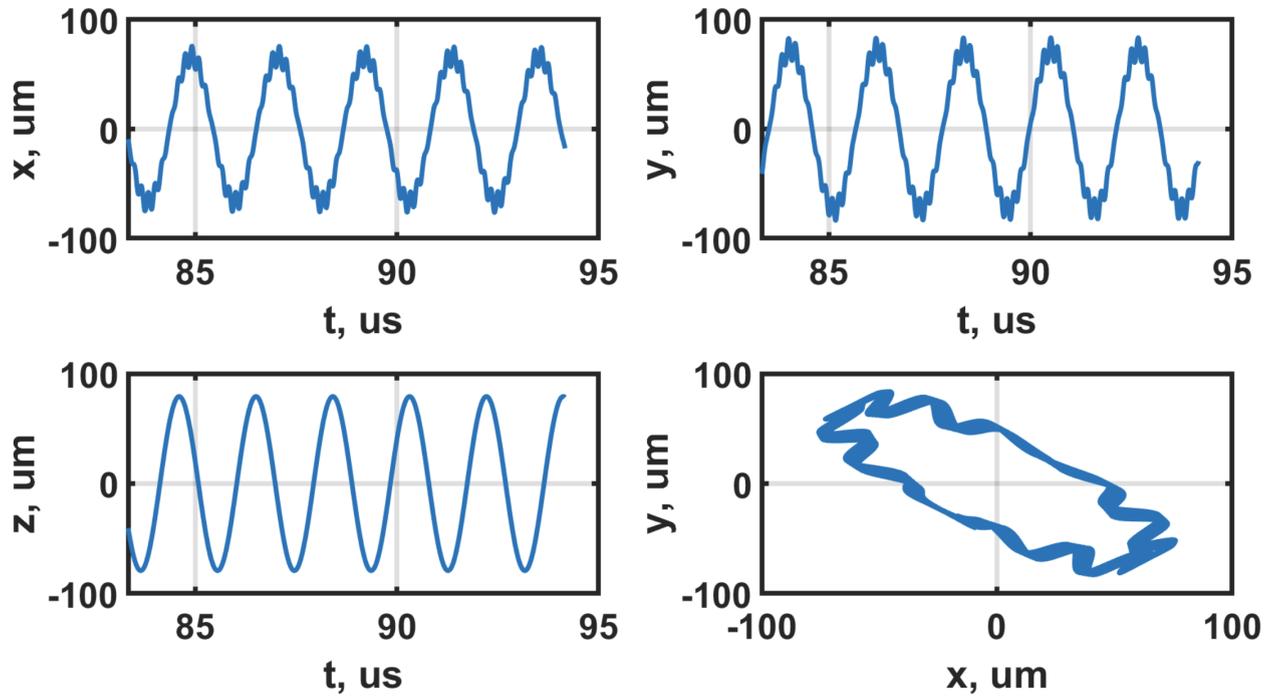
that the frequency of micro-movements also appears in the spectrum, coinciding with the radio frequency supplied to the electrodes.

2.15. *(0.4 points)* Turn on ion cooling. To do this, go to the cell "Langevin bath study in which a viscous friction force is added, acting on the ion and reducing its kinetic energy. The Langevin bath has a cooling rate parameter γ, and a temperature T_0 to which we want to cool (or heat) the ion. Turn on the Langevin bath with different parameters and T_0. Plot the dependence of the ion temperature on time. Compare the time t, during which the ion temperature decreases by 10 times with the parameters of the Langevin bath. As an answer to this point, present graphs and a formula linking the parameters of the Langevin bath with time τ.

2.16. *(0 points)* In the framework of numerical simulation, with the help of a Langevin bath, we can cool the ion to zero Kelvins (we set the parameter $T_0 = 0$). If there are several ions in the trap, then when cooled to zero, they occupy some equilibrium positions in the trap potential and the Coulomb potential of their mutual repulsion, and the resulting "solid"is called a Coulomb crystal. Place 5 ions in the trap at the parameters that you selected in the previous paragraphs, and cool them to zero.

   (a) *(0 points)* Describe the configuration of the resulting crystal?

   (b) *(0 points)* Do the ions line up in a linear chain along the OZ axis? (If the ions are lined up in a linear chain, then the radial coordinates are 0 with very good accuracy, $< 10^{-34}$).

2.17. *(1 point)* The fact whether the ions line up along the line depends on the ratio of the secular frequencies along the radial and axial axes (see the top figure on the next page). Namely, for the linearity of the chain, the trace condition must be met. type:

$$\left(\frac{\omega_r}{\omega_z}\right)^2 \geq \frac{N^\alpha}{A},$$

11

where α and A are some constants that you need to find.
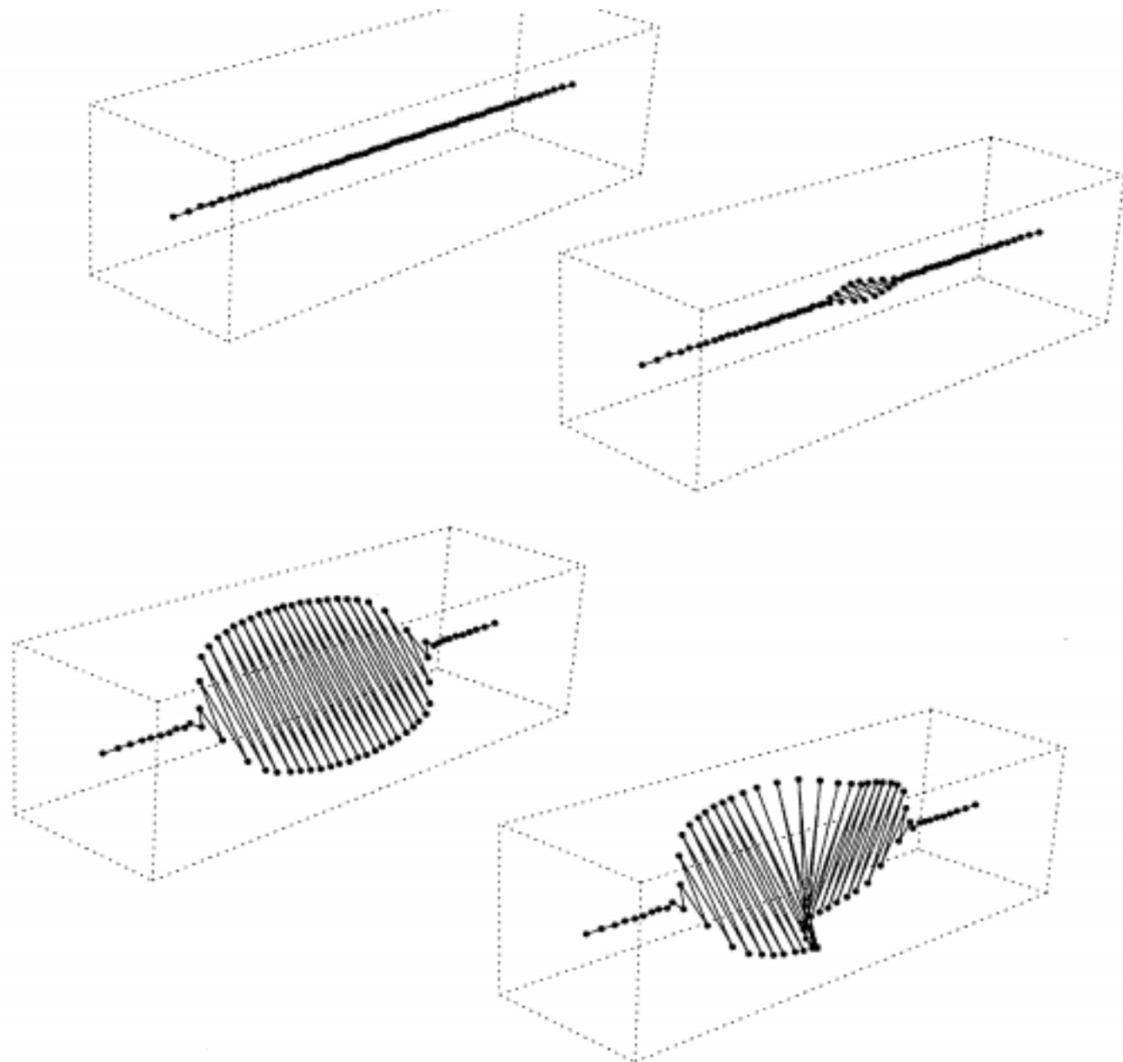
Remark. When conducting simulations at these points, check that you really get a linear crystal, as well as the fact that with these parameters of the trap, ions are captured by it.

At other oscillation frequencies, the ions will line up in a two-dimensional lattice perpendicular to the axial direction (see figure). In this case, the condition must be met:

$$\frac{\omega_z}{\omega_r} > (BN)^\beta.$$

2.20. *(1 point)* find the parameters B and β.
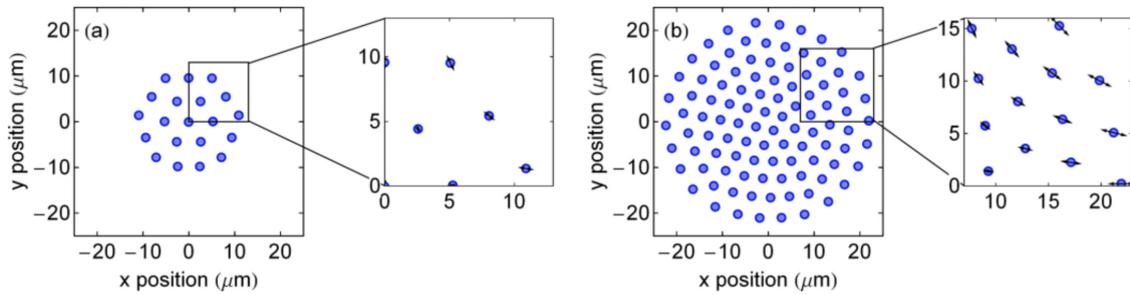
Part 3. (2 points) Optical tweezers

Paul trap is not the only variant of the ion capture scheme. For example, there is a method of optical (laser) tweezers, which will be discussed in this part of the task.

3.1 Geometrical optics approximation

In the case when the wavelength of the radiation is small enough, we can consider the interaction of radiation with matter in the framework of geometric optics. As in Part 2, we will consider light as a stream of photons, the momentum of a single photon will be equal to $\hbar\omega/c$, where ω is the frequency of radiation, and c is the speed of light, $\hbar$ is reduced Planck's constant. Consider that $\hbar\omega$ is the energy of a single photon.

3.1. *(0 points)* Let the light normally fall on a flat surface. The radiation power per unit area is equal to I. What is the light pressure if the surface totally absorbs light? If it totally reflects? If it reflects a fraction of photons equal to α?

3.2. *(0.3 points)* A parallel beam forming a small angle α with the horizon falls on a thin prism with a refractive index n and a small angle Ω at the vertex (see fig.). Find the value and direction of the force acting on the prism from the beam side. Consider that the light is not reflected by the prism. The radiation power per unit area is equal to I. The area of the beam is S. Assume that the whole beam falls on the prism.
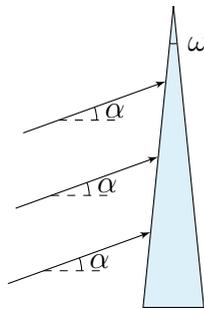
3.3. A thin beam of light falls on a cylinder of radius R with a refractive index n, located in a vacuum, as shown in the figure. The power of light per unit area depends on the coordinate x according to the law $I = I_0(1 - x/R)$ with $x \in [-R/10, R/10]$. At other values of x, the intensity is 0. The height of the cylinder is H.

   (a) *(0.4 points)* What is the force acting on the cylinder?

   (b) *(0.2 points)* Where is it directed?
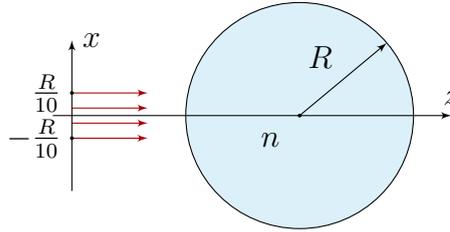
3.4. *(0.1 points)* How will the direction of force change if the refractive index of the cylinder is less than the refractive index of the environment?

Physical optics approximation

If the wavelength is much longer than the object with which the radiation interacts (for example, a calcium ion), the approximation of geometrical optics does not work. Nevertheless, even in this case, the ion may tend to localize in areas where the maxima of the square of the electric field are observed. In this part of the problem, we will consider one of the options for creating such an electric field distribution in the approximation of wave optics.

3.5. *(0.1 points)* A plane electromagnetic wave of frequency ω propagates in vacuum along the optical axis z. In this case, the wave is given by the expression $E_0 \cos((\omega/c)z - \omega t)$. Write down an expression for a wave that propagates against the axis z.

3.6. Suppose an electromagnetic wave of frequency w propagate in vacuum along the optical axis y and normally falls on a conductor located in the plane $z = 0$.
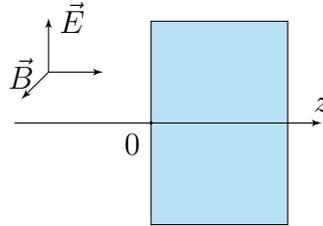
(a) *(0.3 points)* Find the distribution of the electric field in the region $z < 0$ if the wave reflects.

(b) *(0.1 points)* At what distance from the conductor is the first node located in the region $z < 0$ (i.e. the plane in which the oscillation amplitude is zero)?

(c) *(0.1 points)* At what distance from the conductor is the first antinode locate (i.e. the plane in which the oscillation amplitude is maximal)?

3.7. In a real experiment, we use not plane waves, but laser radiation with some intensity distribution in its cross section. As a result, in experiments whose experimental scheme is similar to the previous one, it is possible to obtain the distribution of the electromagnetic field such that the interaction energy of a qubit of mass m and a single charge with an optical trap will have the form:

$$E_{\text{int}} = -C \frac{8P}{\pi w_0^2} \frac{1}{1 + z^2/z_R^2} e^{-\frac{2r^2}{w_0^2\left(1 + z^2/z_R^2\right)}} \cos^2\left(\frac{2\pi}{\lambda_L} z\right),$$

where $C = 377/2\, \alpha_e$ is the polarizability of an atom in units of atomic polarizability, $\alpha_{\text{at}} = 1.65 \cdot 10^{-41}\ C^2 m^2/J$, P is radiation power (from 0 to 10 W), λ is wavelength of

laser radiation ($532\ nm$), w_0 is beam width (50 microns), $z_R = \pi w_0^2/\lambda_L$ is Rayleigh length, $r^2 = x^2 + y^2$ is the distance from the trap axis to the ion.

(a) *(0.2 points)*Find an expression for the force acting along the x-axis.

(b) *(0.2 points)*Find an expression for the force acting along the z-axis.

First hint — 23.05.2023 20:00 (Moscow time)
Second hint — 25.05.2023 20:00 (Moscow time) — *without Alternative problem!*

Final of the fifth round — 28.05.2023 12:00 (Moscow time)