The best hats are cylinder hats.
Tove Jansson. Moominland Midwinter

Cylinder

Ray trajectories in a cylinder

Analogies between different problems in physics with a known solution for one of those problems most of the time allow to find a short solution for the other one.

For example, during the Third Episode of the Second Season of the LPR Cup, 11th graders were asked to obtain the shape of a brachistochrone when a material point moves along a smooth channel inside a homogeneous ball using an optical-mechanical analogy. For this problem, you are also proposed to use the analogy between optics and mechanics. But you need to analyze the trajectory of a ray in an inhomogeneous optical medium.

The basis of geometric optics is Fermat's principle, which states that in an optical medium with a refractive index $n(\vec{r})$ the value of the optical path

$$\ell_0 = \int_A^B n(\vec{r})dl$$

between points A and B takes an extreme value.

The search for equilibrium positions of mechanical systems is based on the principle of potential energy extremum, which states that in the equilibrium position the potential energy of the system takes an extreme value.

Let's consider a weightless string uniformly charged along its length with the density of the charge λ, which is located in an electrostatic field with a potential $\varphi(\vec{r})$. If the string is fixed at points A and B, and its own energy can be neglected, then, from the principle of the potential energy extremum, it follows that the value

$$W_p = \lambda \int_A^B \varphi(\vec{r})dl$$

also takes an extreme value.

Let's consider such a potential that $\varphi(\vec{r}) = An(\vec{r}) + B$, where A is given, and B is an arbitrary constant value. Then, if the lengths of the string and the trajectory of the ray are the same, the ray trajectory and the string form coincide. This analogy can be useful for solving the following problem.

Let's consider an infinitely long cylinder with a radius R and axis z. Its refractive index depends on the distance r to the axis of the cylinder according to the following equation

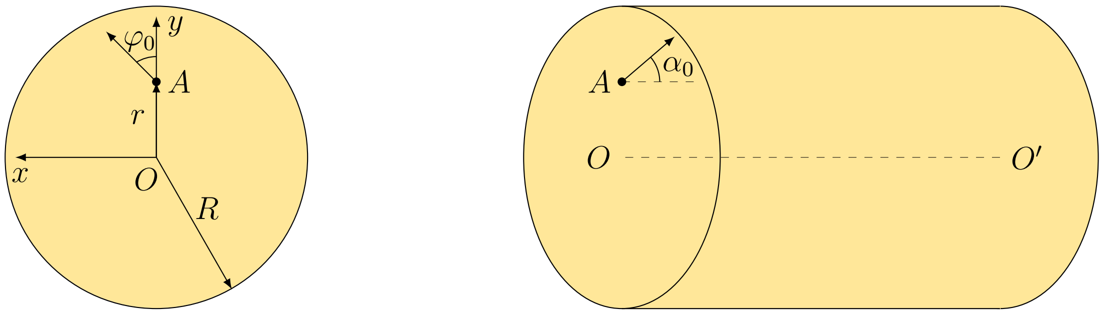
$$n(r) = \sqrt{2 - \frac{r^2}{R^2}}\,.$$

The cylinder is in the air that has a refractive index equal to 1.

We should look at the rays' trajectories that go through point A of the cylinder, which is placed on the distance $r_0 = R/2$ from its axis.

The direction of a ray propagation at the entry point will be characterized by the angle α_0 between the cylinder axis and the wave vector, as well as the angle φ_0, which can be defined as follows. Let $\vec{e}_0$ be a unit vector directed along the ray at point A. Then in the XYZ frame, the vector $\vec{e}_0$ is decomposed as follows

$$(e_{0x}, e_{0y}, e_{0z}) = (\sin\alpha_0 \sin\varphi_0, \sin\alpha_0 \cos\varphi_0, \cos\alpha_0)\,.$$



1. *(3,5 points)* At what value of α_0 is the ray trajectory a helix?

For parts 2 and 3, the value of α_0 is given and equals $\pi/4$.

2. *(4 points)* For an arbitrary value of φ_0, find $r_{\min}$ and $r_{\max}$, the minimum and maximum distance from the points of the trajectory to the axis of the cylinder, respectively. The beam doesn't leave the cylinder.

3. *(2,5 points)* At what values of $\varphi_0 \in [0;\pi]$ does the beam move inside the cylinder without leaving it through the side surface.

First hint — 02.05.2022 14:00 (Moscow time)
Second hint — 04.05.2022 14:00 (Moscow time)

Final of the third round — 06.05.2022 22:00 (Moscow time)