There's a difference between knowing the path and walking the path.
The Matrix (1999)

The Oracle

Introduction

In the Fifth Episode of the Second Season of the LPR Cup, you will be asked to explore diverse optical systems using 2×2 matrices. A video explaining how to work with such matrices is available at link. If you have any questions on how to work with matrices correctly, you can ask Abay.

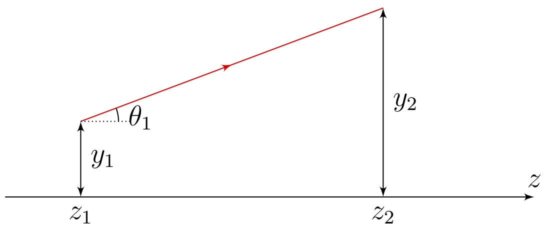
Important! You can ask questions only on how to work with matrices. Questions related to the task should be asked in the private messages of the LPR Cup.

General theory

Let us call an optical system *centered* if the centers of curvature of all spherical refractive and reflecting surfaces are located on a single straight line, which is called *the main optical axis*. If all the beams propagating in the system are at small distances from the optical axis and form small angles with the axis, let us say that the *paraxial approximation* is accurate.

Note. In this problem, unless otherwise specified, let us assume that the paraxial approximation is accurate, and all optical systems are centered.

Let us introduce a Cartesian coordinate system: the Oz axis, which coincides with the main optical axis; Ox and Oy axes, which are perpendicular to the main optical axis, with Oy axis lying in the drawing plane. Consider a beam of rays propagating in the drawing plane. At any point with a known coordinate z, a ray can be uniquely determined if its distance to the optical axis and the angle θ that this ray forms with this axis are known. For example, the figure shows a ray that passes through a point at a distance y_1 from the optical axis and forms an angle θ_1 with this axis (see fig.). Let us measure the angle θ in radians and consider it *positive* if it *corresponds to a counterclockwise rotation* from the positive direction of the z axis to the direction in which the light propagates along the ray.



Although the distance y and the angle θ are obvious parameters for setting the position and direction of the ray propagation, two other parameters are more often used in the literature: the distance y and *optical directional cosine* $v = n \cdot \theta$, where n is the refractive index of the medium at a given point. In the future, let us characterize the ray with this pair of numbers and say that it is unambiguously characterized by the following vector

$$\begin{pmatrix} y \\ n\theta \end{pmatrix} \equiv \begin{pmatrix} y \\ v \end{pmatrix}.$$

When light propagates in an optical system, three processes can occur with a beam: *propagation process*, *refraction of light at the interface of two media*, and *reflection of light*. For each process, let us match $ABCD$ with a matrix by which we will multiply the vector that defines the ray in the plane $z = \text{const}$, as a result, we will get a new vector that corresponds to the new location of the ray. As an example, consider the process of ray propagation in a homogeneous medium.

Matrix of propagation T

The figure above shows the process of ray propagation in a homogeneous medium with a refractive index of n. Consider two planes with coordinates z_1 and z_2. It is clear that the angle between the ray and the main optical axis in both planes is the same, so

$$\theta_2 = \theta_1 \quad \Longleftrightarrow \quad v_2 = v_1,$$

where $v_1 = n\theta_1$ and $v_2 = n\theta_2$.

On the other hand, the coordinate y_2 can easily be written in terms of y_1 and θ_1. Indeed:

$$y_2 = y_1 + \operatorname{tg}\theta_1(z_2 - z_1) \approx y_1 + \theta_1(z_2 - z_1) = y_1 + v_1\frac{z_2 - z_1}{n}.$$

From the last two equations, we can get that the equation of ray propagation in a homogeneous medium can be written as

$$\begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{z_2 - z_1}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix}.$$

And the $ABCD$ matrix of propagation is

$$T = \begin{pmatrix} 1 & \dfrac{z_2 - z_1}{n} \\ 0 & 1 \end{pmatrix}$$

If the ray is involved in several processes in a row, some transformations should be done with it. These transformations are equivalent to matrices multiplication. Indeed, if the ray was at a distance y_1 from the optical axis and propagated at a distance l_1 along it, this is equivalent to multiplying the vector with the components y_1 and v_1 by the corresponding matrix of propagation T_1

$$\begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{l_1}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix}.$$

If the ray continued to propagate in a homogeneous medium for an additional distance l_2, then this is equivalent to multiplying the vector with the components y_2 and v_2 by the matrix T_2

$$\begin{pmatrix} y_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{l_2}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{l_2}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{l_1}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = T_2 T_1 \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = T \begin{pmatrix} y_1 \\ v_1 \end{pmatrix}.$$

Thus, we can say the final transformation matrix T is equal to the product of two propagation matrices written in the **reverse** order. There we used the fact that the products of matrices are associative. So, the following statement is true:

$$ABC = (AB)C = A(BC)$$

As an exercise, make sure that the matrix T has the form

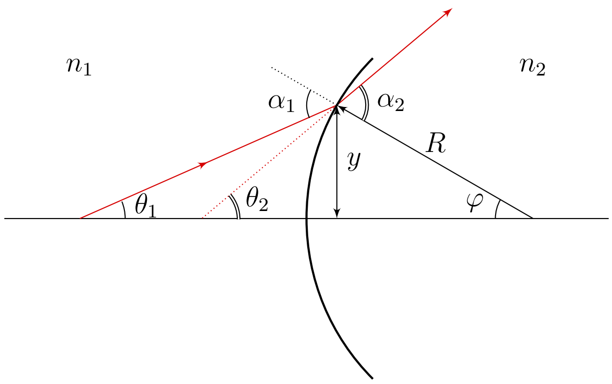
$$\begin{pmatrix} 1 & \dfrac{l_1 + l_2}{n} \\ 0 & 1 \end{pmatrix}.$$

Note that in this case the relation $T_1 \times T_2 = T_2 \times T_1$ is true. In other words, the matrices commute, which is not always true, including the examples that we will discuss later. Therefore, the order of writing the matrices is very significant! And in our case, the matrices are written in the **reverse order**!

Matrix of refraction P

Consider a spherical interface between two media with refractive indexes n_1 and n_2. Let the radius of curvature of the surface be positive if the angle between the axis Oz and the radius-vector which connects the center of curvature and the spherical surface is obtuse. If this angle is acute, then let this radius of curvature be negative (see fig.).

Let us consider the refraction of light on a spherical surface and find the matrix of refraction P. Let the ray pass from a medium with a refractive index n_1 to a medium with a refractive index n_2 (see fig.).



It is clear that the y coordinate does not change when the ray crosses the interface between the two media, so

$$y_2 = y_1.$$

Let the angles of incidence and refraction be α_1 and α_2, respectively, and the angles between the optical axis and the incident and refracted rays – θ_1 and θ_2. The figure shows that $\alpha_1 = \theta_1 + \varphi$, and $\alpha_2 = \theta_2 + \varphi$, where φ is the angle between the optical axis and the radius to the point where the ray is refracted. Let us write the Snell's law $n_1\alpha_1 = n_2\alpha_2$ and use the fact that $\varphi = y/R$, then

$$n_1 \left(\theta_1 + \frac{y}{R}\right) = n_2 \left(\theta_2 + \frac{y}{R}\right).$$

Rewriting the last equation in terms of the directional cosine v_1 and v_2, we get that

$$v_2 = \frac{n_1 - n_2}{R} y_1 + v_1,$$

and then we find

$$\begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{n_2 - n_1}{R} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix}.$$

In the lower-left corner of the matrix, let us take out a sign and select a fraction, which is called the optical power of the surface P_1

$$P_1 = \frac{n_2 - n_1}{R}.$$

Thus, we get that the matrix of refraction has the form

$$P = \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix}.$$

Exercise 1. Show that a thin biconvex lens with radii of curvature $R_1 > 0$ and $R_2 < 0$ and a refractive index n, placed in a medium with a refractive index n_0, has the following matrix of transformation of optical rays

$$\begin{pmatrix} 1 & 0 \\ -(P_1 + P_2) & 1 \end{pmatrix},$$

where $P_1 + P_2 = (n - n_0)\left(\dfrac{1}{R_1} - \dfrac{1}{R_2}\right) = \dfrac{1}{F}$.

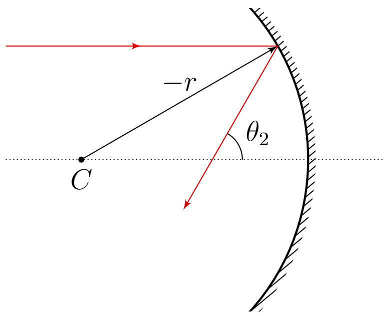
Exercise 2. Find the optical power of a thin biconvex lens with radii of curvature $R_1 > 0$ and $R_2 < 0$ and a refractive index n, if it is placed between two media with refractive indexes n_1 and n_2.

Exercise 3. Prove that the optical powers of two lenses which are close to each other add up.

Matrix of reflection R

Find the reflection matrix for a spherical mirror. A beam running parallel to the axis of a mirror (see fig.) is given by the vector

$$\begin{pmatrix} y \\ 0 \end{pmatrix}.$$



After reflection, it passes through the focus of the spherical mirror located at a distance $r/2$ from its vertex. Immediately after reflection, the beam height does not change, and the slope angle of the beam is

$$\alpha = -\frac{y}{-r/2}.$$

Here we took into account that the angle and curvature radius are negative, so the reflected beam is characterized by the vector

$$\begin{pmatrix} y \\ \frac{2}{r}y \end{pmatrix}.$$

The incident and reflected rays are connected by the reflection matrix R

$$\begin{pmatrix} y \\ \frac{2}{r}y \end{pmatrix} = R\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} Ay \\ Cy \end{pmatrix}.$$

It follows from the equality of the vectors coordinates that

$$A = 1; \quad C = \frac{2}{r}.$$

Now let's reverse the direction of the ray just discussed. Then the ray before reflection is given by the vector

$$\begin{pmatrix} y \\ -\frac{2}{r}y \end{pmatrix}.$$

After reflection, it is given by the vector

$$\begin{pmatrix} y \\ 0 \end{pmatrix}.$$

The incident and reflected rays are still connected by the reflection matrix R

$$\begin{pmatrix} y \\ 0 \end{pmatrix} = R\begin{pmatrix} y \\ -\frac{2}{r}y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} y \\ -\frac{2}{r}y \end{pmatrix} = \begin{pmatrix} Ay - B\frac{2}{r}y \\ Cy - D\frac{2}{r}y \end{pmatrix}.$$

Equating the coordinates of the vectors, we get

$$B = 0; \quad C - D\frac{2}{r} = 0.$$

From which (taking into account the expression for C) we find

$$D = 1.$$

All elements of the reflection matrix are found

$$R = \begin{pmatrix} 1 & 0 \\ \frac{2}{r} & 1 \end{pmatrix}.$$

Note. When considering reflections, one of the additional assumptions is usually made:

- The positive direction of the z axis is taken along the beam path.
- When the direction of the beam is changed, the direction of the z axis remains the same, while the refractive index of the medium changes by $-n$.

Problem

Part 1

Let there be some optical system, which is described by some $ABCD$-matrix that transforms a ray outgoing from the plane with coordinate z_1 into a ray entering the plane z_2. The parameters of the optical system were selected so that one of the matrix elements became equal to zero. What physical property does the system have if

1. $A = 0$;
2. $B = 0$;
3. $C = 0$;
4. $D = 0$.

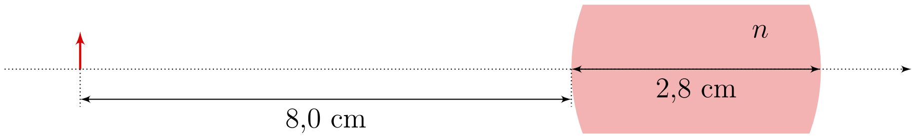
Note. Each of the points weighs zero points, but you can send them, and they will be checked in the `CPI` format so you can make the right conclusions from your reasoning.

Part 2

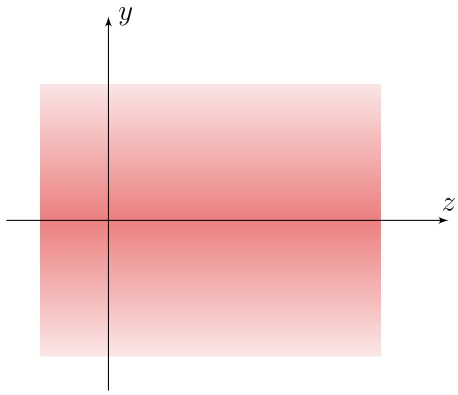
5. *(0,5 points)* The eyepiece of the Hedgehog telescope consists of two thin positive lenses with optical powers P_1 and P_2 made of the same material and located at some distance from each other. At what distance between lenses a dependence of the refractive index of glass on wavelengths will not affect the optical power of the eyepiece? Consider the wavelength being placed in a small spectral interval in the surrounding area of a wavelength λ_0.

Part 3

6. *(0,5 points)* Both ends of a glass cylindrical rod 2,8 cm long have a spherical shape with a radius of 2,4 cm. Refractive index of the glass is 1,6. An object in the form of a straight line 0,5 cm long is placed on the axis of the rod in a vacuum at a distance of 8,0 cm from the left end of the rod. Find the position and size of the image.



Part 4



Consider a plate with a refractive index which depends only on the distance to the optical axis according to the law

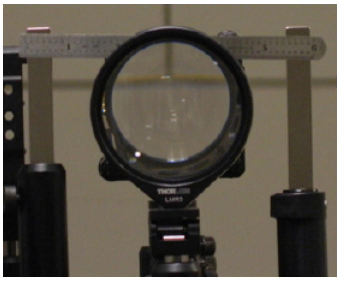
$$n(y) = n_0 - n_1 \frac{y^2}{2}$$

The width of the plate is L, and the thickness is a. Consider the constants n_0 and n_1 known, it is also known that $n_1 a^2 \ll n_0$.

7. *(1 point)* Find $ABCD$-matrix of the plate.

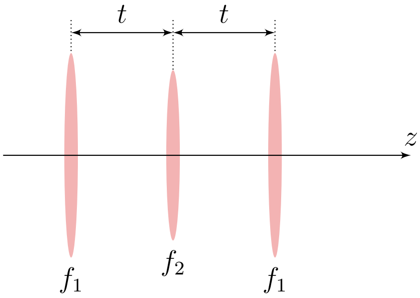
Part 5

It is known that for certain parameters of the lens system, objects located on the periphery of the space between the lenses become invisible, and the images of objects outside the optical system are not distorted, as if there were no optical system (see fig.).

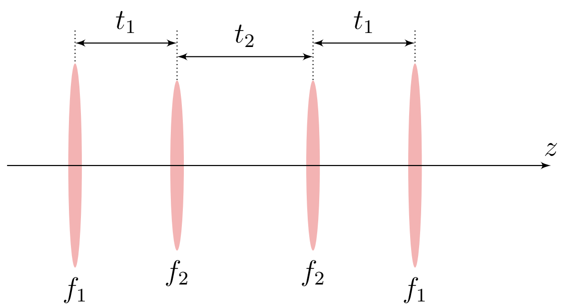


Note. In all items of this problem there is **no need to prove** the existance of the invisibility area.

8. *(1 point)* Show that symmetrical system of three thin lenses with focal lengths distances f_1, f_2, and f_1, respectively (see fig. below) satisfies the above-described condition only if $f_1 \gg t$, where t is a distance between lenses.



9. *(3 points)* Find the ratio between f_1 and f_2 focal lengths for a system of four thin lenses with focal lengths f_1, f_2, f_2, and f_1 respectively (see fig. below), at which this phenomenon will be observed. Determine at what ratio f_1/f_2 the length of the optical system reaches the extremum. What is the ratio t_2/f_2? Consider the distance between the first and second lenses being equal to the distance between the third and fourth lenses.

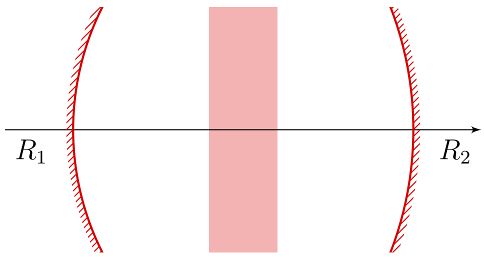


Note. In all the tasks, the distance between lenses and their focal lengths are unknown. The chromatic aberration can be neglected.

Part 6

The figure shows a resonator consisting of 2 spherical mirrors with radii R_1 and R_2 at a distance L from each other, and some optical element with a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$



Moreover, it is known that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 1.$$

System stability

10. *(1 point)* Find the ray transmission matrix for one «cycle» of the ray in the resonator.

Let us denote the matrix obtained in task 1 by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

11. *(2 points)* Find the equations on A, B, C, and D corresponding to the trajectory lines of the ray that does not leave the resonator through a large number of reflections. The answer should be expressed in terms of A, B, C, D. **Note.** Correct can also be obtained with the wrong task 10.

Chaotic behavior of light

At this point consider

$$\begin{cases} R_1 = -1 \text{ m}; \\ R_2 = 2 \text{ m}; \\ L = 1{,}001 \text{ m}. \end{cases}$$

The optical element matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

12. *(1 point)* For rays

$$\begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_2 \\ 0 \end{pmatrix}$$

such that $|y_1 - y_2| \ll |y_1|$, find the rate of divergence. Determine the nature of dependence (linear, exponential, polynomial, etc.).

Note. The rate of divergence is the function of the distance between the rays depending on the number of resonator passes. Its dimension is meters per number of resonator flights.

Instruction. Perhaps somewhere in the problem, it will be convenient to find some solutions in the following form

$$x_k = x_{\max} \sin\left(k\omega + \varphi_0\right);$$

$$x_k = x_{\max} \operatorname{sh}\left(k\omega + \varphi_0\right).$$

Here $k \in \mathbb{N}$, $\operatorname{sh} x = \dfrac{e^x - e^{-x}}{2}$ is a hyperbolic sine.

First hint — 31.05.2021 14:00 (GMT-2)
Second hint — 02.06.2021 14:00 (GMT-2)

End of the fifth tour — 04.06.2021 22:00 (GMT+3)