LPR VI Cup

10.s06.e05

Any ending is the beginning of something new.
Folklore

Contents

Whole

In the Second Episode for grades 9-10, we talked with the LPR Cup Participants about time, and in grade 11 we offered a problem that could be elegantly solved using a geometric method. In the Third Episode for grades 10-11, Participants were introduced to the variational principle and the use of symmetry to analyze the propagation of light in gradient media.

~~This bzzzz is not for nothing~~ All of this was not for nothing.

When studying objects that move at high speeds (at speeds comparable to the speed of light), the classical laws of Newtonian mechanics become inapplicable and yield results that contradict experiment. To describe the motion of such objects, a separate branch of physics was developed, called the *special relativity theory*. As experimental data continued to accumulate, particularly in the study of gravity, it became clear that the theory could be developed further, leading to the creation of the *general relativity theory*.

These branches of physics speak the language of geometry and use concepts such as: metric, interval, curvature, and so on. In addition, these fields use the concepts of action and the variational principle.

In this Episode, we will tell you about this special language, how it works, and what results can already be obtained and understood with its help.

In section 1, we will discuss issues related to the special theory of relativity (SRT), after which we will see that SRT is not sufficient to describe objects such as black holes (section 2), and we will invite Participants in grades 10-11 to move on to the general relativity theory (section 3).

Good luck!

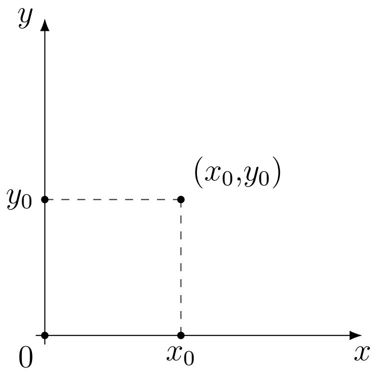
Charles the Cat

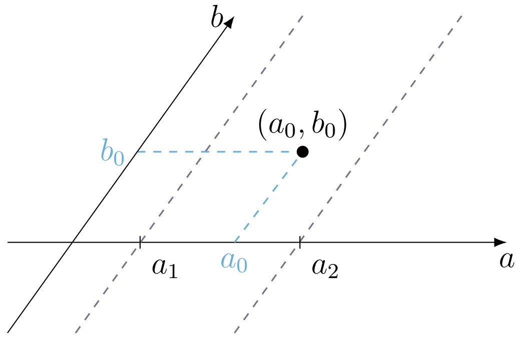
1 Kinematics of SRT

The special relativity theory (SRT) is a physical theory that has found a huge number of practical applications and actively uses the language of geometry — specifically, the geometry of Minkowski spacetime. In this section, we will study the kinematic aspects of SRT.

Calculating Distances in Different Coordinates

Let us consider a two-dimensional Cartesian coordinate system. In this case, the position of points is specified by a pair of numbers (x, y). To find the coordinate of a point along one of the axes by construction, we need to draw a straight line through this point parallel to the other axis until it intersects the first axis. This algorithm is general and also works in the case where the axes are not perpendicular to each other. It also follows that the lines of constant coordinate along one axis are straight lines parallel to the other axis.





A trajectory in such a coordinate system is defined as the dependence of one coordinate on another, for example, $y(x)$. The distance in such a coordinate system is determined using the Pythagorean theorem, that is, a small element of length is written as:

$$ds^2 = dx^2 + dy^2.$$

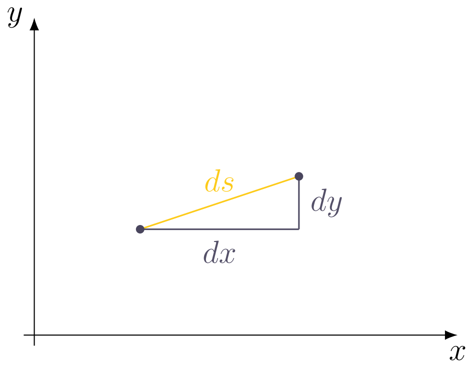
Note that in the case of non-perpendicular axes, the distance is determined by the law of cosines:

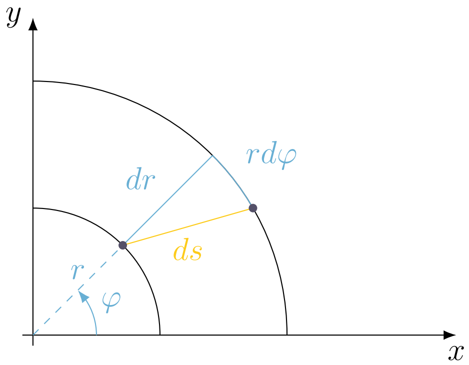
$$ds^2 = da^2 + db^2 + 2da db \cos\alpha,$$

where α is the angle between the axes a and b.

The Cartesian coordinate system is not the only way to specify the position of a point on a plane. There are many other options, for example, polar coordinates, in which the position of a point is given by the distance from the origin r and the angle φ measured from some axis (see the figure). When switching to polar coordinates, the distance takes the form:

$$ds^2 = dr^2 + r^2 d\varphi^2.$$





Postulates of SRT

The special relativity theory of is based on the following postulates:

1. The laws of physics are the same in all inertial reference frames.

2. The speed of light in vacuum is constant.

The first postulate is straightforward—we assume that processes in all inertial reference frames (IRFs) occur in the same way, and therefore the physical laws are written in the same form.

The second postulate, however, seems quite surprising, but it is precisely this postulate that serves as the starting point for constructing SRT. The fact that the speed of light is constant was experimentally confirmed in various experiments conducted in the second half of the 19th century (the Fizeau experiment, the Michelson–Morley experiment, and others).

Galilean and Lorentz Transformations

In this section, we will consider the transition from one inertial reference frame K to another inertial reference frame K'.

Let us consider the motion of a material point along the x-axis with velocity v. When moving from the laboratory reference frame to the reference frame of this point, the coordinates and time are transformed according to the law

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t.$$

These transformations are called Galilean transformations. From them, it follows that the velocities in this reference frame are

$$v'_x = \frac{dx'}{dt'} = v_x - v, \quad v'_y = \frac{dy'}{dt'} = v_y, \quad v'_z = \frac{dz'}{dt} = v_z.$$

Such a law of velocity transformation contradicts the second postulate of special relativity. Let us confirm this with a classic example. Imagine a person running with a flashlight in hand. In the reference frame associated with the running person, photons (or the electromagnetic wave) have a speed equal to the speed of light in vacuum (hereafter denoted as c, whose numerical value we will take to be approximately $3 \cdot 10^8$ m/s). In the reference frame associated with a stationary observer, according to the Galilean transformations, light should have a speed of $c + u$, where u is the speed of the running person, which contradicts the second postulate of special relativity, according to which this speed must be equal to c.

Let us find the transformation of coordinates and time under which the speed of light does not change when moving from one inertial reference frame to another (for convenience, we will call these frames K and K'). Since in any inertial reference frame the dependence $x(t)$ for a point moving at constant speed is linear, these transformations must map straight lines to straight lines, i.e., they must be linear. In the general case, such transformations have the form:

$$\begin{cases} x' = Ax + Bt, \\ t' = Cx + Dt. \end{cases}$$

1.1. *(0 points)* Using the fact that the dependence of the photon's coordinate on time in two different inertial reference frames K and K' is written as $x = ct$ and $x' = ct'$, find the explicit form of the transformations that preserve the speed of light (these are called Lorentz transformations):

$$\begin{cases} x' = \gamma\left(x - vt\right), \\ t' = \gamma\left(t - \dfrac{v}{c^2}x\right), \end{cases}$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ is the relativistic "gamma factor" and v — velocity of K' in relatively to K.

Note 1. In this example, we considered motion only along the x-axis, so the coordinates along the y and z axes will have the form: $y' = y$, $z' = z$.

Note 2. It is clear that in the classical (non-relativistic) limit, when $v/c \ll 1$, the Lorentz transformations coincide with the Galilean transformations.

Note 3. From the expression for the "gamma factor," it follows that a reference frame moving at a speed greater than the speed of light is not acceptable in this theory and has no physical meaning. Thus, we can conclude that the speed of light in vacuum is not just a constant, but the maximum allowed speed for any particle.

Using the fact that when transforming back from K' to K only the sign of the velocity v changes, it is clear that the inverse Lorentz transformation has the form:

$$\begin{cases} x = \gamma\left(x' + vt'\right), \\ t = \gamma\left(t' + \dfrac{v}{c^2}x'\right), \end{cases}$$

1.2. *(0 points)* Category Hobbit There and Back Again. Make sure that by consecutively applying two Lorentz transformations for the transitions $K \to K' \to K$, we obtain the identity transformation for coordinates and time.

Let us consider the motion of the same material point along the x-axis in two inertial reference frames K and K'. Writing the Lorentz transformation for two moments in time and subtracting one system of equations from the other, we get:

$$\begin{cases} dx' = \gamma\left(dx - vdt\right), \\ dt' = \gamma\left(dt - \dfrac{v}{c^2}dx\right), \\ dy' = dy, \\ dz' = dz. \end{cases}$$

Minkowski Spacetime

Now that we have established that under Lorentz transformations, time—just like spatial coordinates—transforms nontrivially, it makes sense to combine them into a single Minkowski spacetime, where the coordinates of points are given by (ct, x, y, z).

1.3. *(0.2 points)* Draw and describe how, under Lorentz transformations for motion only along the x-axis, the coordinate axes ct and x change.

In Minkowski spacetime, each point is an *event* that occurs at a certain spatial coordinate at a certain moment in time. The distance between two events in Minkowski spacetime is called the *interval* and is calculated as follows:

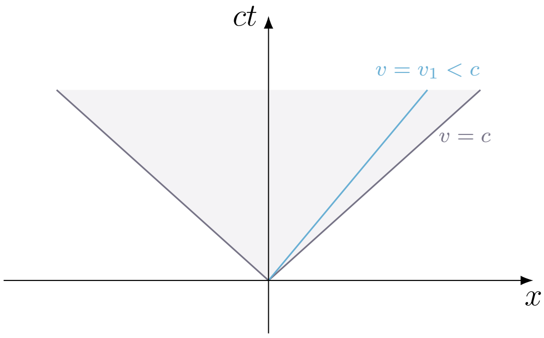
$$ds^2 = g_{tt}c^2dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

The set of coefficients g_{tt}, g_{xx}, g_{yy}, g_{zz} is called the *metric.*

1.4. *(0.2 points)* Show that Lorentz transformations do not change the interval of Minkowski spacetime (such transformations are called isometries of the metric).

The square of the interval between two events can have any sign. Let us consider the events (0,0) and (ct, x) and the interval ds between them.

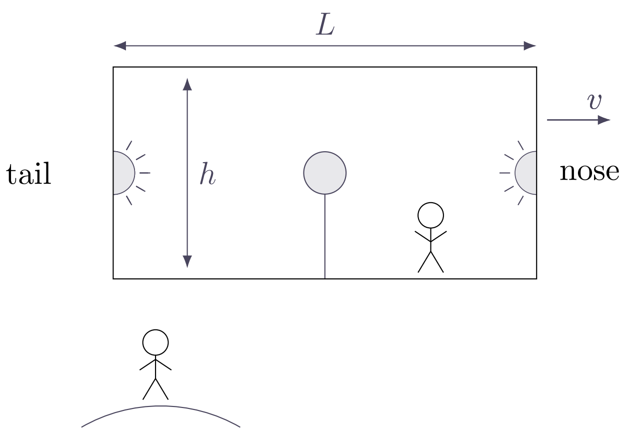
- Events for which $ds^2 = 0$ lie on the light cone. These events correspond to a photon emitted from $(0,0)$ arriving at the point with spatial coordinate x at time t. Such an interval ds is called *lightlike* (or *null*).
- Events for which $ds^2 < 0$ lie outside the light cone. Such an interval is called *spacelike*.
- Events for which $ds^2 > 0$ lie inside the light cone. Such an interval is called *timelike*. Particles in Minkowski spacetime move along curves for which, for any small segment, $ds^2 > 0$. Such curves are called *worldlines*.



Effects of Relativistic Kinematics

Let us consider a few examples that will demonstrate the peculiarities of Lorentz transformations.

1.5. *(0.3 points)* Suppose a rocket is flying through outer space at a speed $v = c/2$. The interior cabin of the rocket has the shape of a rectangular parallelepiped, with light sources installed at the centers of the "nose" and "tail." Short flashes from these sources occur simultaneously according to the clock of an astronaut inside the rocket. Find the difference between the detection times of the flashes by a receiver located at the center of the cabin, as measured in the reference frame of an observer on a stationary planet. In the same reference frame, find the difference between the emission times of flashes by the sources. The distance from the "nose" to the "tail" in the reference frame of a rocket is L.



1.6. *(0.3 points)* Depict the events of light detection and emission in the reference frames of an astronaut and observer on the planet using Minkowski spacetime coordinates.

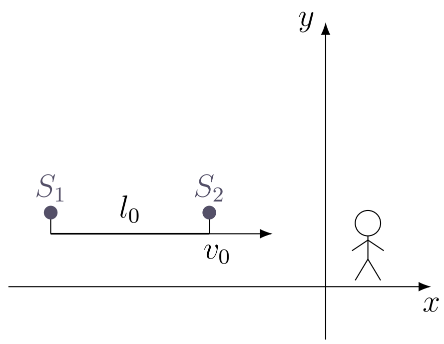
1.7. *(0.3 points)* For these two observers, how are the times related for the light emitted from the "tail" source to first reach the "floor" of the rocket, if the distance from the source to the ceiling is h?

Now let us consider a real-life example from the world of particle physics.

1.8. *(0.3 points)* A meson is produced in the atmosphere at an altitude of about 10 km above the Earth's surface. Its proper lifetime (proper quantities for a given object are those measured in its own rest frame) is approximately 2,2 μs. Estimate what speed the meson must have in order to reach a detector on the Earth's surface.

And now a few more abstract examples.

1.9. *(0.5 points)* A rod has a proper length l_0. Two light bulbs, S_1 and S_2, are attached to the ends of the rod. The rod moves with velocity v_0 toward a stationary observer. Bulb S_1 emits light earlier than S_2, so that both flashes reach the observer simultaneously. At the moments the light is emitted, bulbs S_1 and S_2 are located at points x_1 and x_2, respectively. What distance $x_1 - x_2$ between the bulbs will the observer measure? (This will be the apparent length of the rod, as perceived by the human eye or recorded by a camera.)



1.10. *(0.4 points)* From the origin of the inertial reference frame K, short light pulses are sent along the x-axis at intervals of time T (according to the clocks in K). Find the time interval between the moments these signals are registered by an observer if the observer is in the inertial reference frame K', which is moving towards system K with velocity $v = c/2$.

The Twin Paradox

1.11. *(0.2 points)* A spaceship leaves Earth at speed βc at time $t_0 < 0$ (by earthly clocks), where β is a known constant. The captain's twin brother remains on Earth. At time $t_0/2$ by earthly clocks, the spaceship makes a quick turnaround and returns to Earth at the same speed as before. Analyze the astronaut's journey in the reference frame of the Earth observer and show that the astronaut's proper time elapsed during the trip is less than the proper time of his brother (in other words, the twin astronaut has aged less).

Note that similar reasoning can be made from the point of view of the traveling brother, who considers himself at rest and the Earth observer as moving relative to him at the same speed. In this case, it would seem that the brother on Earth has aged less. Explain qualitatively how to resolve this paradox, and approximately what the actual difference in proper times of the two twins will be (indicate which one will be older at the end of the journey).

1.12. *(0.8 points)* A spaceship leaves Earth at speed $v = \beta c$ at time $t_0 < 0$ (by Earth's clock), where β is a known constant. The captain's twin brother remains on Earth. At time $t_0/2$ (by Earth's clock) he observes a light signal from the spaceship that means that it begins to turn around. The worldline of the spaceship during the turnaround is a segment of a hyperbola (in reference frame of the Earth):

$$\frac{(x - x_0)^2}{a^2} - \frac{c^2 t^2}{a^2} = 1.$$

Here, the parameters a and x_0 are unknown, and c is the speed of light. After the turnaround, the spaceship moves at a constant speed $-v$. Find by how much the ages of the twins will differ when the spaceship returns to Earth. Assume that the speed of the spaceship changes continuously along its entire worldline. You may neglect the segments of the worldline where the spaceship accelerates and decelerates near Earth.

You may find the following integral useful:

$$\int_a^b \frac{dx}{\sqrt{x^2+1}} = \ln\left(\frac{b+\sqrt{b^2+1}}{a+\sqrt{a^2+1}}\right).$$

Momentum and Energy

Let us consider four-dimensional Minkowski spacetime with coordinates (ct, x, y, z). For uniformity of notation, these coordinates can be combined into a single 4-vector:

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \mathbf{r} \end{pmatrix}.$$

As mentioned earlier, the proper time for a system moving at velocity v is the time $d\tau$ measured by a clock at rest in that system. Using the Lorentz transformation, we can obtain the connection between $d\tau$ and dt, where dt is measured by a clock in the stationary frame:

$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} = \gamma d\tau.$$

Taking dt outside the square root in the expression for the interval, we get:

$$ds = \sqrt{c^2 dt^2 - dr^2} = c dt \sqrt{1 - v^2/c^2} = c d\tau,$$

where $dr^2 = dx^2 + dy^2 + dz^2$ is the spatial part of the interval. Thus, we have found a very convenient connection: the interval, which is the analog of distance in spacetime, can be parameterized by the proper time of the particle.

Let us define the 4-velocity vector as follows:

$$U = \frac{d}{d\tau}\begin{pmatrix} ct \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} c dt/d\tau \\ d\mathbf{r}/d\tau \end{pmatrix} = \gamma \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix},$$

and also the 4-momentum of the particle:

$$P = mU = \begin{pmatrix} \gamma mc \\ \gamma m\mathbf{v} \end{pmatrix}.$$

Let us consider the expression for the time component of the 4-momentum and transform it for the case of small velocities ($v/c \ll 1$):

$$P_t = \frac{mc}{\sqrt{1 - v^2/c^2}} = mc + \frac{mv^2}{2c} + \ldots \approx \frac{1}{c}\left(mc^2 + \frac{mv^2}{2}\right).$$

The expression in parentheses coincides with the kinetic energy of *massive particles* with a certain fixed addition. Therefore, we say that this is the definition of the relativistic energy of such particles ($E = \gamma mc^2$):

$$P_t = E/c \quad \Longrightarrow \quad P = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}.$$

1.13. *(0 points)* By analogy with the interval, find the "length" of the 4-momentum.

1.14. *(0 points)* There exist particles whose mass m is zero. They are called *massless*. Using the result from the previous item, express the energy E of such particles in terms of their momentum.

2 Classical black hole

In the previous section, you became acquainted with the postulates of the Special Theory of Relativity. It turns out that one of its direct consequences is the possibility of the existence of special objects—black holes—which inevitably "absorb" any matter that comes sufficiently close to them. In this section, you are invited to conduct your own investigation of such an object.

Through prolonged observation of the position of a star near the center of the galaxy, it was found that the star undergoes periodic motion in the gravitational field of a certain massive object. Assume that the distance from the observation point to the center of the galaxy is approximately $26 \cdot 10^3$ light-years, the period of the star's orbit along its trajectory is approximately 16 years, and the plane in which the star moves is perpendicular to the line of sight.

2.1. *(0.3 points)* Using the experimental data, determine the possible positions of the object attracting the star in the coordinates provided in the problem.

2.2. *(0.4 points)* Using the experimental data, determine the mass of the object attracting the star.

2.3. *(0.8 points)* Assuming the object attracting the star is spherically symmetric and its size is sufficiently small, determine the boundaries of the region around such an object from which no signal can reach a distant observer.

Table 1: Coordinates of the star observation

$x,'' \cdot 10^3$	−36	−34	−31	−27	−21	−15	−8	−1	9	17	24	32	39
$y,'' \cdot 10^3$	122	132	143	153	164	172	178	182	185	184	181	175	167

The experimental data are given in an orthogonal coordinate system, where each axis is measured in arcseconds ($''$) and represents the angular deviation of a point from the origin.

You are allowed to use numerical methods to analyze the experimental data, and the parameters determined from them must be obtained with an accuracy of $5 \cdot 10^{-3}('')$.

3 GRT

The Theoretical Minimum

The general theory of relativity (GTR) is currently the most precise theory of classical gravity. The main idea of GTR lies in the fact the the gravitational interaction is nothing, but the consequence of the spacetime curvature."Curvature" means that the metric in GTR g[1] depends on the point in spacetime:

$$ds^2_{\text{OTO}} = g_{tt}(\mathbf{r},t)dt^2 + \sum_{i=1}^{3} 2g_{ti}(\mathbf{r},t)dtdx^i + \sum_{i,j=1}^{3} g_{ij}(\mathbf{r},t)dx^i dx^j \tag{1}$$

unlike the special relativity theory, where

$$ds^2_{\text{CTO}} = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{2}$$

Note that if the coordinates are indexed with the index i, dx^i does not mean dx to the power of i, but the infinitesimal change of the i-th coordinate. For example, $(x, y, z) = (x^1, x^2, x^3)$, then if $i = 1$, $dx^i = dx$, if $i = 2$, $dx^i = dy$, if $i = 3$, $dx^i = dz$ [2]. Further we write, for instance, g_{rr} as if $i = r$, which means that this is a multiplier factor for dr^2. Here dr^2 means the second power of dr.

The metric g can be found from the Einstein's equation. For the full picture we provide it here:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{3}$$

The right side of this equation is determined by the distribution of the mass and its motion. The left side is calculated from the metric g.

For convenience, we use the system of units, where the speed of light in vacuum is unity: $c = 1$. This means that the dimensions of dt and dx are the same, and the proper time $d\tau$ coincides with the length of the world line ds.

We are interested in the motion of the probe particles in vacuum in the gravitational field of the spherically symmetrical object of mass M, which is located in the start of the coordinates. Such an object might be a planet, a star, a black hole. Since we consider motion in vacuum and neglect the gravitational field of particles, the right side of the equation (3) is zero area of spacetime we are interested in. Because of the spherical symmetry of the massive object, we are looking for the spherically-symmetrical solution of the equation (3), which is called the Schwarzschild solution. It turns out that it is unique and has the following form:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2(d\theta^2 + \sin^2\theta \cdot d\varphi^2). \tag{4}$$

The parameter r_g is called the Schwarzschild's radius. As we will see further, it is determined only by the mass M.

[1] By g we mean a set of functions g_{tt}, g_{tx}, g_{ty}, g_{tz}, g_{xx}, g_{yy}, g_{zz}, g_{xy}, g_{xz}, g_{yz}. In spherical coordinates the indices (r, φ, θ) will be instead of (x, y, z)

[2] In the Cartesian coordinates (x, y, z) the squared spacetime interval equals $ds^2 = g_{tt}(\mathbf{r},t)dt^2 + 2g_{tx}(\mathbf{r},t)dxdt + 2g_{ty}(\mathbf{r},t)dydt +$
$+ 2g_{tz}(\mathbf{r},t)dzdt + g_{xx}(\mathbf{r},t)dx^2 + g_{yy}(\mathbf{r},t)dy^2 + g_{zz}(\mathbf{r},t)dz^2 + 2g_{xy}(\mathbf{r},t)dxdy + 2g_{xz}(\mathbf{r},t)dxdz + 2g_{yz}(\mathbf{r},t)dydz.$

3.1. *(0 points)* Find g_{tt}, g_{rr}, $g_{\theta\theta}$, $g_{\varphi\varphi}$, g_{tr}, $g_{t\theta}$, $g_{t\varphi}$, $g_{r\theta}$, $g_{r\varphi}$, $g_{\theta\varphi}$ in the notations of the equation (1) for the metric (4).

If angles θ and φ are constant, i. e. if we are interested only in the radial motion, in other words the motion in the plane (t, r), this solution simplifies:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)}. \tag{5}$$

In GTR for the description of the particles motion the laws, which are analogous to the Fermat's principle, are used. The principle of Fermat describes the motion of light in space with the given refractive index $n(x,y)$. Let us describe these laws.

The motion of a free particle of mass m in the curved spacetime is defined by the principle of the least action: a particle moves from point A to point B along the world line γ, which minimizes[3] the integral, called *action* S_γ[4]:

$$S_\gamma = -\int\limits_\gamma m\, ds. \tag{6}$$

It is clear that the action of the free particle is the length of the world line γ in the spacetime with metric g up to a multiplier.

Such a way to determine the world line (trajectory) of a particle should remind one of the Fermat's principle in the ray optics (see. the 3rd Episode 10-11 grades). Indeed, the squared optical path equals

$$ds^2 = n^2 dl^2 = n(x, y)^2 dx^2 + n(x, y)^2 dy^2.$$

It coincides with (1) in two space dimensions. Meanwhile, all components g_{ij} are the same and equal to n^2.

From (6) one can obtain the equations for $\mathbf{r}(\tau)$, $t(\tau)$. The solutions of these equations are called *geodesics*. It can be shown that there are three types of geodesic lines:

- *Timelike geodesics*: in any point $ds^2 > 0$;
- *Lightlike geodesics*: in any point $ds^2 = 0$;
- *Spacelike geodesics*: in any point $ds^2 < 0$;

Timelike geodesics are the world lines for massive particles. We are also interested in the lightlike geodesics (as one can easily guess, the light follows these geodesics). They can be obtained in the limit $m \to 0$ from timelike geodesics. It corresponds to the fact, that the light particles, photons, have a zero mass m.

However, we will not solve the geodesics equations explicitly. We will find the world lines of the particles, using the action's (6) symmetries, instead.

[3] or maximizes

[4] The capital letter S_γ will always denote the action on the curve γ, and the lowercase letter s_γ — length γ.

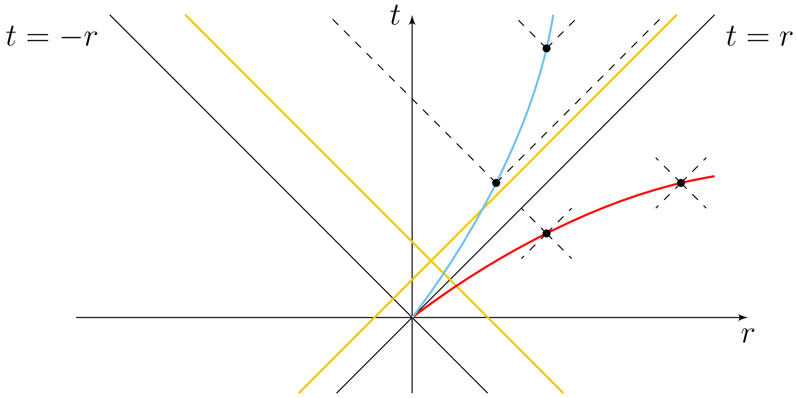
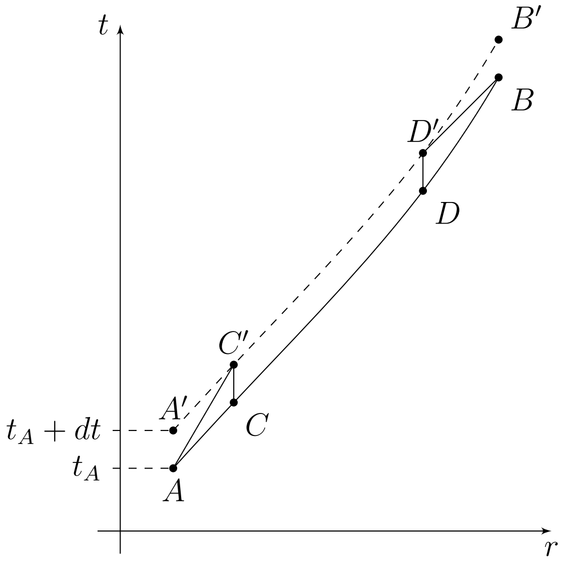


Figure 1: Lightlike, timelike и spacelike geodesics in the Minkovskii's space (2).

Motion in the plane (t, r)

Let us study the motion of a particle of given mass m in the plane (t, r) in the spacetime with metric (5) with the given Schwarzschild's radius r_g. Let $ACDB$ be a world line of a particle, and a curve $A'C'D'B'$ is obtained from it by the shift on dt, i. e. $t_{A'} - t_A = t_{B'} - t_B = t_{C'} - t_C = t_{D'} - t_D = dt$ (see the picture). Consider the coordinates of the points A, B, C, D as known, as well as the fact $dt \ll s_{AC}/(t_C - t_A), s_{BD}/(t_B - t_D)$.



Let us choose the points C, A and B, D close to each other, so that we can assume AC, BD are segments. It means, for instance, that $S_{AC} = -m\Delta s_{AC} = -m\sqrt{g_{tt}(t_C - t_A)^2 + g_{rr}(r_C - r_A)^2}$.

3.2. *(0.2 points)* Find the approximation[5] of $S_{AC'} - S_{AC}$ in the first order with respect to dt.

3.3. *(0.2 points)* Find the approximation of $S_{BD'} - S_{BD}$ in the first order with respect to dt.

3.4. *(0.2 points)* Find $S_{CD} - S_{C'D'}$.

3.5. *(0 points)* Since $ACDB$ minimizes (6), the first order approximation of $S_{ACDB} - S_{AC'D'B}$ should be zero. Derive the conservation law from this condition.

As you known from the special relativity theory, the relativistic expression for the energy of

[5] see 0-th Hint for the 3rd Episode in the 10-11th Grages

a particle has the following form

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = mc^2 \frac{cdt}{\sqrt{c^2dt^2 - dx^2}}.$$

The expression in this form can be generalized for the GRT case:

$$E = \frac{mg_{tt}dt}{\sqrt{g_{tt}dt^2 + g_{rr}dr^2}} = mg_{tt}\frac{dt}{ds}. \tag{7}$$

From 3.5 it is clear, the energy defined this way is conserved, i. e. does not depend on the point, where it is calculated. Let us check the correctness of the limit transitions. To do this, consider the case of small $v_0 = dr/dt$ and r_g/r_0. For this case we find:

$$E(v_0, r_0) \approx m + \frac{mv_0^2}{2} - \alpha\frac{r_g}{r_0}. \tag{8}$$

The obtained expression (8) must coincide with the energy of a particle in the gravitational field of a body with mass M (known from Newton's theory of gravity) up to an additive constant.

3.6. *(0.1 points)* Find α, decomposing (7) to the first order of v_0^2 and r_g/r_0. Comparing the result with Newton's result, write r_g in terms of the mass M and the gravitation constant G.

Let us solve (7) with respect to t:

$$E = mg_{tt}\frac{dt}{ds} \quad\Longrightarrow\quad m^2g_{tt}^2dt^2 = E^2(g_{tt}dt^2 + g_{rr}dr^2), \quad\Longrightarrow\quad dt = \pm\sqrt{-\frac{g_{rr}}{g_{tt}}}\frac{dr}{\sqrt{1 - \frac{m^2}{E^2}g_{tt}}}.$$

We obtain the equation for the world line $t(r, E)$:

$$t(r, E) - t_0 = \pm\int\limits_{r_0}^{r} \frac{dr}{\left(1 - \frac{r_g}{r}\right)\sqrt{1 - \frac{m^2}{E^2}\left(1 - \frac{r_g}{r}\right)}}. \tag{9}$$

Falling into the Black Hole

Let the considered massive object be a black hole. It means that the mass M is concentrated inside the *event horizon* $r = r_g$, meanwhile its distribution inside the horizon is unknown, but we assume that all the mass M is located in the center of coordinates. Consequently we may use[6] the obtained equation for the world line (9) $r \geqslant r_g$, maybe at $r < r_g$. Let us study the properties of the obtained world lines (9). For the convenience we write the equation for the velocity of a particle:

$$v = \frac{dr}{dt} = \pm\left(1 - \frac{r_g}{r}\right)\sqrt{1 - \frac{m^2}{E^2}\left(1 - \frac{r_g}{r}\right)}. \tag{10}$$

Consider a particle that moves away from the black hole in the region of $r > r_g$ (sign «+» in expression (10)).

3.7. *(0.1 points)* At which E will the particle not reach the infinity and turn round?

[6]Reminder: the metric (1) was obtained for vacuum, i. e. when there is no mass is in the area of motion.

3.8. *(0.2 points)* Let the energy of a particle E so that it does not reach the infinity. What is the maximal distance $r_{\max}$?

Now consider the motion of the particle after it turned round (sign «$-$» in expression (10)). In such motion r decreases, becoming closer to r_g. To understand, what happens to $t(r,E)$ at $r \to r_g$, we need the following mathematical statement:

$$\int_{x_0}^{x} \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\frac{ax+b}{ax_0+b}, \qquad \int_{x_0}^{x} \frac{1}{x^\alpha}\,dx = \frac{1}{(\alpha-1)x^{\alpha-1}} - \frac{1}{(\alpha-1)x_0^{\alpha-1}}, \quad \text{where} \quad \alpha \neq 1. \tag{11}$$

Here we see that the first integral is diverging (i. e. goes to infinity) when $x \to -b/a$, but the second integral has a finite limit $1/((\alpha-1)x_0^{\alpha-1})$ at $\alpha < 1$.

From this mathematical statement, one can find that the time of falling onto the horizon $t(r_g,E)$ (see (9)) is infinite. It is true even for the lightlike geodesics, i. e. when $m = 0$. For this reason, the outside observer never sees that the particle has reached the event horizon of a black hole. Does it mean that the particle will not reach it? To address this question, we need to calculate the proper time of the particle:

3.9. *(0.2 points)* Find the change of the proper time of a particle $\tau(r,E) - \tau_0$.

3.10. *(0 points)* Make sure that $\tau(r_g,E)$ is finite.

Due to the fact that $\tau(r_g,E)$ is finite, the particle will reach the event horizon and go through it. Since the time $t(r_g,E)$ is infinite, we need to change the frame of reference to describe the particle's world line, i. e. change the coordinate system. Let us choose a new coordinate for the time: $(t,r) \to (x^+,r)$, where

$$x^+ = \begin{cases} t + r + r_g \ln\left(\dfrac{r}{r_g} - 1\right), & r > r_g \\ t + r + r_g \ln\left(1 - \dfrac{r}{r_g}\right), & r < r_g. \end{cases} \tag{12}$$

3.11. *(0.2 points)* Express ds^2 (see equation (5)) in terms of dx^+ and dr. For this purpose express dt from the equation (12) in terms of dx^+ and dr.

3.12. *(0.5 points)* Find the equations of the lightlike geodesics in the coordinates (x^+,r).

3.13. *(0.3 points)* Sketch the graphs of all types of the lightlike geodesics in one plane (t',r), $t' = x^+ - r$.

In each point of a plane (t',r) two lightlike geodesics intersect. The world lines of particles should lie inside a light cone (in the region where $ds^2 > 0$) with a top at point (t',r). From the obtained results one can see that no particle, even light, cannot go beyond the event horizon from within it.

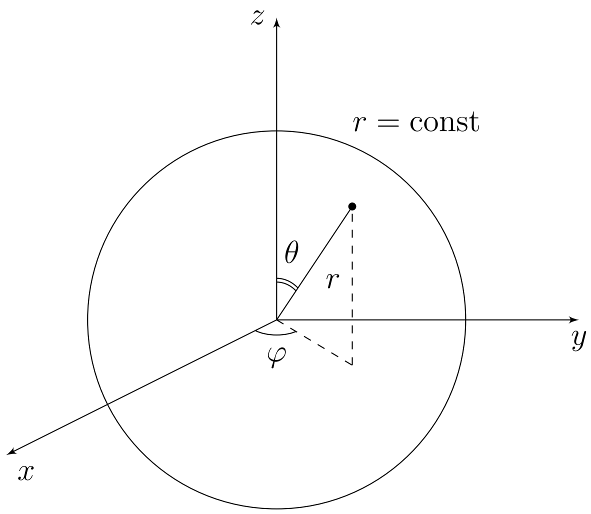
Finally, let us note that from the result for the proper time, obtained in question 3.9, one can see that $\tau(0,E)$ is finite. It means that the particle reaches the singularity in the center of a black hole at a finite proper time.

The general case of a particle motion the Schwarzschild metric

Now let us consider the general case of a particle (with mass m) moving in the metric (4). As it was, the corresponding action (6) remains unchanged under shifts of time, so the energy is still conserved:

$$E = mg_{tt}\frac{dt}{ds} = \text{const}\,. \tag{13}$$

The action (6) also has a spherical symmetry. Due to it the motion of a particle in space (x, y, z) is happening in one plane. Without loss of generality, we can assume that the motion occurs in a plane $(x, y, 0)$.



In terms of coordintes (r, θ, φ) it means that $\theta = \pi/2$. Then the metric (4) simplifies:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 d\varphi^2. \tag{14}$$

The action with such metric has only a part of the spherical symmetry left: the symmetry that corresponds to the rotations around the normal to the plane of motion of the particle. In terms of coordinates it means the invariance under angle changes $\varphi \to \varphi + d\varphi$.

3.14. *(0.6 points)* Derive a conservation law corresponding to this symmetry (for example, perform a similar reasoning to 3.2-3.5).

To fix the normalization, we remind that in the analogous problem of the Newtonian gravity the law of angular momentum conservation J_{H} is present. To define the angular momentum, we divide the velocity vector into two components — radial and azimuthal:

$$v^2 = v_r^2 + v_\varphi^2, \quad \text{where} \quad v_r = \frac{dr}{dt}, \quad v_\varphi = r\frac{d\varphi}{dt}.$$

Then $J_{\text{H}} = mrv_\varphi$. Let us denote the obtained conserved quantity (3.14) as J. We choose the normalization J to obtain J_{H} in the limit of small r_g/r and v:

$$J = mr^2\frac{d\varphi}{ds}.$$

To determine the world line of a particle, t, φ, r and parameters E, J need to be related. We will parameterize the world line through r, then the equation of the world line is $t(r, E, J)$, $\varphi(r, E, J)$. Consider the mass of a particle m, the Schwarzschild radius r_g and initial coordinate r_0 to be known.

3.15. *(0.1 points)* Find $t(r, E, J) - t_0$.

3.16. *(0.1 points)* Find $\varphi(r, E, J) - \varphi_0$.

In both questions the expression should be in the form of an integral, which are analogous to (9).

Let us analyze the obtained results.

3.17. *(0.3 points)* How many types of the world lines are there when $J = 0.9mr_g$? Describe the particle motion on such world lines. Explain your answer.

3.18. *(0.3 points)* How many types of the world lines are there when $J = 1.9mr_g$? Describe the particle motion on such world lines. Explain your answer.

3.19. *(0.3 points)* How many types of the world lines are there when $J = 2.9mr_g$? Describe the particle motion on such world lines. Explain your answer.

3.20. *(0.3 points)* Find the boundary values $J/(mr_g)$, by passing which the possible types of trajectories change, with the precision not worse than 0.01.

Perihelion precession of Mercury

Let us recall how the motion of a point in the gravitational field of a body with mass M occurs in Newtonian theory. The equation of motion can be found using the law of conservation of energy:

$$E_{\mathrm{N}} = \frac{mv^2}{2} - \frac{GmM}{r} = \mathrm{const}\,.$$

Since, in addition to energy, the angular momentum $J_{\mathrm{N}} = mrv_\varphi = \mathrm{const}$ is also conserved, it is convenient to rewrite the energy conservation law in terms of J_{N}:

$$E_{\mathrm{N}} = \frac{mv_r^2}{2} - \frac{GmM}{r} + \frac{J_{\mathrm{N}}^2}{2mr^2} = \mathrm{const}\,.$$

Using these two conservation laws, it is easy to obtain the equation of the world line in the Newtonian approximation:

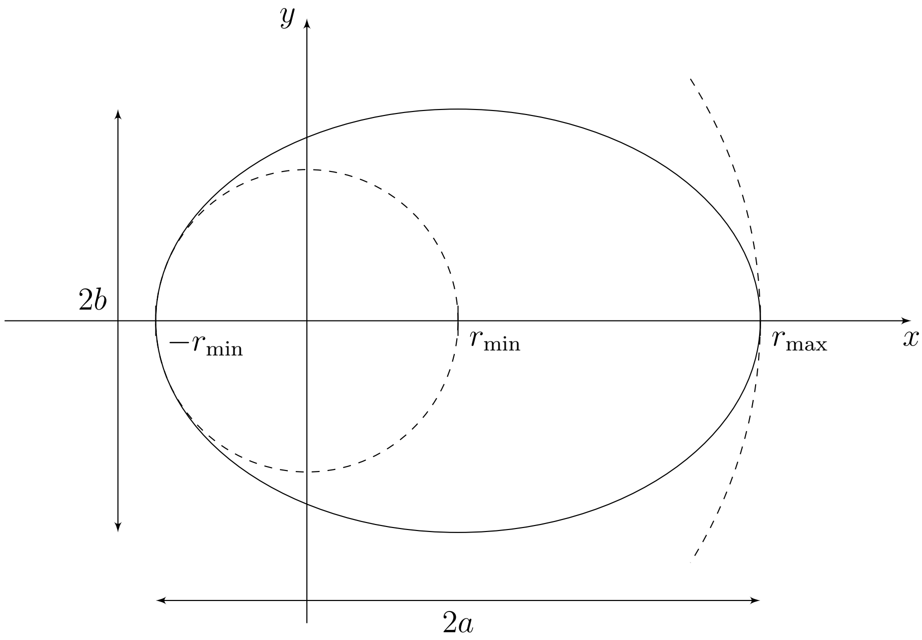
$$t_{\mathrm{N}}(r, E_{\mathrm{N}}, J_{\mathrm{N}}) - t_0 = \pm m \int\limits_{r_0}^{r} \frac{dr}{\sqrt{2mE_{\mathrm{N}} + \frac{2Gm^2M}{r} - \frac{J_{\mathrm{N}}^2}{r^2}}} \tag{15}$$

$$\varphi_{\mathrm{N}}(r, E_{\mathrm{N}}, J_{\mathrm{N}}) - \varphi_0 = \pm J_{\mathrm{N}} \int\limits_{r_0}^{r} \frac{1}{r^2} \frac{dr}{\sqrt{2mE_{\mathrm{N}} + \frac{2Gm^2M}{r} - \frac{J_{\mathrm{N}}^2}{r^2}}} \tag{16}$$

As is known from Kepler's laws (and this can also be easily derived from (16)), if $E_{\mathrm{N}} < 0$, the equation in (16) describes an ellipse with a focus at the origin. The equation of such an ellipse has the form

$$r_{\mathrm{N}}(\varphi) = \frac{p_{\mathrm{N}}}{1 - e_{\mathrm{N}} \cos\varphi},$$

where $e_{\mathrm{N}} = \sqrt{1 - b^2/a^2}$ is the eccentricity, $p_{\mathrm{N}} = b^2/a$, a is the semi-major axis, and b is the semi-minor axis (see Wikipedia).



From (16) it is easy to see that

$$e = \sqrt{1 + \frac{2E_N J_N^2}{G^2 M^2 m^3}}, \quad p = \frac{J_N^2}{Gm^2 M}, \quad a = -\frac{GmM}{2E_N}, \quad b = \frac{J_N}{\sqrt{-2E_N m}}.$$

Equation (16) has the form

$$\varphi_N(r, E_N, J_N) - \varphi_0 = \pm J_N \int_{r_0}^{r} \frac{1}{r^2} \frac{dr}{\sqrt{f_N(r, E_N, J_N)}}, \quad \text{where} \quad f_N(r, E_N, J_N) = 2mE_N + \frac{2Gm^2 M}{r} - \frac{J_N^2}{r^2}.$$

If the point makes one complete revolution along the trajectory, i. e., r changes from $r_{\min}$ to $r_{\max}$ and back, then the angle changes by 2π:

$$2J_N \int_{r_{\min}}^{r_{\max}} \frac{1}{r^2} \frac{dr}{\sqrt{f_N(r, E_N, J_N)}} = 2\pi. \qquad (17)$$

The equation obtained in section 3.16 has a similar form:

$$\varphi(r, E, J) - \varphi_0 = \pm J \int_{r_0}^{r} \frac{1}{r^2} \frac{dr}{\sqrt{f(r, E, J)}}.$$

We want to obtain the correction to the result (17) from general relativity. To do this, let us consider the motion of a particle with energy $E = m + \mathcal{E}$ and angular momentum J. We will assume that

$$\frac{|\mathcal{E}|}{m} \sim \frac{J^2}{m^2 r^2} \sim v^2 \sim \frac{r_g}{r} \ll 1, \qquad (18)$$

i. e., these are small quantities of the same order.

It can be shown that, to second order (i. e., up to squares and products of the small quantities in (18)), the following equality holds:

$$f(m + \mathcal{E}, J, r) = f_N(\tilde{\mathcal{E}}, \tilde{J}, \tilde{r}), \qquad (19)$$

where $\tilde{\mathcal{E}} = \mathcal{E}(1+\delta_\mathcal{E})$, $\tilde{J}^2 = J^2(1-\delta_J)$, $\tilde{r} = r+\delta_r$. The corrections $\delta_\mathcal{E}$, δ_J, and δ_r do not depend on r, and are small quantities of the order $\delta_\mathcal{E} \sim \delta_J/J^2 \sim \delta_r/r \sim \mathcal{E}/m$. They can be found by expanding the difference $f(m+\mathcal{E}, J, r) - f_\text{N}(\tilde{\mathcal{E}}, \tilde{J}, \tilde{r})$ to second order in smallness and equating to zero all remaining coefficients at powers of r.

3.21. *(0.2 points)* Find δ_E.

3.22. *(0.2 points)* Find δ_J.

3.23. *(0.2 points)* Find δ_r.

Using (19), we can find the correction of interest:

$$\Delta\varphi_\text{gr} - 2\pi = 2\int\limits_{r_\text{gr,min}}^{r_\text{gr,max}} \frac{J}{r^2}\frac{dr}{\sqrt{f(r, m+\mathcal{E}, J)}} - 2\int\limits_{\tilde{r}_\text{min}}^{\tilde{r}_\text{max}} \frac{\tilde{J}}{\tilde{r}^2}\frac{d\tilde{r}}{\sqrt{f_\text{N}(\tilde{r}, m+\tilde{\mathcal{E}}, \tilde{J})}} \approx$$

$$\approx 2\int\limits_{\tilde{r}_\text{min}}^{\tilde{r}_\text{max}} \left(\frac{J}{r^2} - \frac{\tilde{J}}{\tilde{r}^2}\right)\frac{d\tilde{r}}{\sqrt{f_\text{N}(\tilde{r}, m+\tilde{\mathcal{E}}, \tilde{J})}}.$$

Expanding the difference in the round brackets, we get

$$\frac{J}{r^2} - \frac{\tilde{J}}{\tilde{r}^2} \approx \frac{\tilde{J}}{\tilde{r}^2}\left(\frac{r_g}{\tilde{r}} - 2A + 3A\right),$$

where A does not depend on r. It can be shown that the integral of the term with $r_g/r - 2A$ does not contribute.

3.24. *(0 points)* If you know how to perform variable substitutions in integrals, show this using the substitution $r \to \xi = r_g/r - 2A$.

Then, using (17), we obtain the answer:

$$\Delta\varphi_\text{gr} - 2\pi = 2\pi \cdot 3A.$$

3.25. *(0.2 points)* Express A in terms of m, r_g, and J.

3.26. *(0 points)* Express A in terms of the geometric parameters of the orbit and obtain the numerical value of $\Delta\varphi_\text{gr} - 2\pi$ for Mercury.

First hint — 28.05.2025 20:00 (Moscow time)
Second hint — 30.05.2025 12:00 (Moscow time)

Final of the fifth round — 01.05.2025 18:00 (Moscow time)